A NEW APPROACH TO AN OLD PROBLEM

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There can be few students of Electrical Engineering who have not at some time or other come across the so-called anomaly of an apparent loss of energy when a charged capacitor is connected across an uncharged one. The writer considers the usual explanations unsatisfactory and attempts here a different approach.

The Problem

Initially capacitor Ct (Fig. 1) has a charge \( Q \), and the energy stored in \( C_t \) is \( \frac{1}{2}Q^2/C_t \). When the switch \( S \) is closed there is a redistribution of charge, and since there is no loss of charge the final stored energy is \( \frac{1}{2}Q^2/C_t + \frac{1}{2}Q^2/C_t = \frac{1}{2}Q^2/C_t \) and there is apparently a loss of \( \frac{1}{2}Q^2/C_t \). The problem is, of course, to account for the loss.

The usual explanation, in terms of resistance and inductance of connecting wires, losses in the capacitors and even of electromagnetic radiation, is irrelevant. It would be reasonable enough if the problem were phrased in these terms; but it is not. The circuit as illustrated in Fig. 1 shows that no mention has been made of inductance or resistance and certainly none of electromagnetic radiation. We must concern ourselves only with the circuit as shown, which is idealized in accordance with usual practice.

The Solution

In order to investigate the problem the following artifice is adopted: a resistance \( R \) is inserted in series with the capacitors (Fig 2).

As before \( C_1=C_2 \) and initially \( C_1 \) is charged to \( Q \). After an interval \( t \) after the switch \( S \) is closed, the charge on \( C_2 \) is \( q \). We have therefore

\[
R \frac{dq}{dt} + q/c = \frac{Q - q}{C} \quad \text{and} \quad R \frac{dq}{dt} + \frac{2q}{C} = \frac{Q}{C} \quad (i)
\]

The solution of this is

\[
q = (Q/2) \left( 1 - e^{-2t/CR} \right) \quad \text{and} \quad i = (dq/dt) = (Q/CR) e^{-2t/CR} \quad \text{and} \quad (ii)
\]

In an interval \( dt \) the energy dissipated in \( R \) is

\[
P_R dt = (Q^2/CR) e^{-2t/CR} \]

and the total energy dissipated in \( R \) is

\[
\left( \frac{Q^2}{CR} \right) \int_0^\infty e^{-2t/CR} dt = \frac{Q^2}{4C} \quad \text{..... (iv)}
\]

Returning to the original problem, we require an expression for \( q \), the charge on \( C_2 \), when \( R = 0 \). We therefore let \( R \) tend to zero in equation (ii)

\[
\text{i.e. } q = \lim_{R \to 0} \left( \frac{Q}{2} \left( 1 - e^{-2t/CR} \right) \right) \quad \text{(v)}
\]

\[
\text{i.e. } q = \lim_{R \to 0} \left( \frac{Q}{2} \left[ 1 - \frac{2t}{CR} + \frac{4t^2}{C^2R^2} - \frac{8t^3}{C^3R^3} + \cdots \right] \right)
\]

\[
\text{i.e. } q = \lim_{R \to 0} \left( \frac{Q}{2} \left[ 1 - \frac{2t}{CR} + \frac{4t^2}{C^2R^2} \right] + \frac{8t^3}{C^3R^3} + \frac{1}{3!} - \cdots \right)
\]

\[
\text{i.e. } q \to (Q/2) \left[ -\infty - \infty + \infty + \cdots \right] \text{as } R \to 0 \quad (vi)
\]

Conclusions

The physical interpretation of equation (vi), if one exists, is difficult since \( q \) is expressed in terms of an oscillating series whose successive terms are plus and minus infinity for finite values of \( t \), and which becomes indeterminate for infinite values of \( t \). It is clear however that no steady state solution exists for the case when \( R = 0 \). Herein lies the fallacy of the original problem; writing down a steady state solution to give the charges on \( C_1 \) and \( C_2 \) when in fact no steady state exists.

For the case when \( R > 0 \) the steady state solution for the charges on the two capacitors is given by equation (ii) whence \( q = Q/2 \) when \( t = \infty \).

From equation (iv) it will be seen that the energy dissipated in \( R \) is \( Q^2/4C \), i.e. half the initial energy, and that this value is independent of the value of \( R \) for \( R > 0 \).

An alternative approach is to include inductance and resistance initially in the connecting wire. The result of closing the switch is a damped, sinusoidal oscillation of charge between the two capacitors. If \( R \) is now put equal to zero the damping is removed and the oscillation is maintained. The period of oscillation is a function of \( L \) and tends to zero as \( L \) tends to zero. The limiting case when \( L = 0 \) corresponds to the original problem confirming that there is no steady state when \( R = 0 \) and \( L = 0 \).