Decentralised controller design for a class of interconnected systems

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Abstract: This study considers the decentralised controller design for interconnected systems. For the case that the interconnections satisfy certain matching conditions, a decentralised static controller is presented, which is designed in a separate way. On the basis of the presented controller, a relaxed sufficient condition is proposed, and the corresponding decentralised dynamic controller is designed for interconnected systems with two subsystems satisfying the relaxed sufficient condition. The efficiency of proposed controllers is demonstrated both theoretically and numerically.

1 Introduction

The decentralised control has attracted widespread attention since the 1960s, due to the wide applications of large-scale systems [1-4]. The significant feature of decentralised control is that each subsystem generates a controller with its own state or input information to stabilise the whole system. Owing to the structure constraint of the controller, it is difficult to develop necessary and sufficient conditions for decentralised control of large-scale systems. Wang and Davison [5] presented a necessary and sufficient condition for the decentralised dynamic controller of large-scale systems. However, the controllers designed in [5] are of a high order with a step-by-step method, which makes it difficult to implement in practice. Therefore, the sufficient conditions and the necessary conditions are always studied separately. The literature about the necessary conditions of large-scale systems is mainly based on the fixed mode [6, 7]. While for the study of sufficient conditions, results from different perspectives are proposed in the existing literatures [1, 2, 8-12]. Sufficient conditions for large-scale systems with special structures are studied in [1, 2, 8, 9], where [1, 2] consider the case that the interconnections satisfy some certain matching conditions, while symmetric interconnections are conducted in [8, 9]. Decentralised H∞ problem is studied in [10]. Other control theories or techniques are also used to propose sufficient conditions for decentralised control of large-scale systems such as small gain theorem [11] and large gain theorem [12].

Despite these studies on necessary or sufficient conditions for decentralised control of large-scale systems, another difficult problem is how to design decentralised controller. It should be noted that the decentralised controller designed in [5] is a step-by-step method, i.e. the controller design of the i-th subsystem is based on the controllers of previous i-1 subsystems. Such drawback of the decentralised controller also exists in [1, 2, 11], since as the number of subsystems grows, the time delay of the controller will also increase and the stabilisation of the whole large-scale system will be broken. In view of this, some linear matrix inequality (LMI)-based decentralised controllers are proposed by using the robust control method [10, 13]. A common feature of these controllers is that it requires a common matrix satisfying every LMIs of each subsystem. Therefore, such controller design method is of high conservativeness. Recently, iterative algorithm for decentralised control is developed in [14]. However, such an algorithm cannot ensure a satisfactory decentralised controller for large-scale systems.

This paper revisits the decentralised control problem of large-scale systems. The rest of this paper is organised as follows. In Section 2, based on the sufficient conditions that the interconnections satisfy matching conditions (as is presented in [1]), an LMI-based decentralised controller is proposed. Different from the decentralised controllers designed in [1] with a step-by-step method, the controller presented in this paper is in a separate way that each subsystem designs its own controller without using that of other subsystems. Furthermore, based on the theoretical proof of the effectiveness of the proposed controller, a less conservative sufficient condition is given. Decentralised controller for the presented sufficient condition is also proposed. In Section 3, for the special case that the system composed of two subsystems, the decentralised dynamic controller is presented, and the relations with disturbance rejection are also discussed. Simulation results are presented in Section 4 and Section 5 concludes this paper.

2 Decentralised static controller design

Consider the following interconnected system of N subsystems with state interconnections

\[ x_i = A_i x_i + \sum_{j \neq i} B_{ij} u_j + B_{ik} u_k, \quad i = 1, \ldots, N, \quad (1) \]

where \( x_i \in \mathbb{R}^{n_i} \) and \( u_i \in \mathbb{R}^{m_i} \) are the state and control inputs of the i-th subsystem, \( A_i \) and \( B_i \) are the dynamic and input matrices of the i-th subsystem, and \( A_{ij} \) is the interconnection matrix from the j-th subsystem to the i-th subsystem.

The model of the large-scale system (1) can be rewritten in a compact form as

\[ \dot{x} = Ax + Bu, \quad (2) \]

where \( x = [x_1^T, \ldots, x_N^T]^T \) is the state vector and \( u = [u_1^T, \ldots, u_N^T]^T \) is the control input vector

\[ A = \begin{bmatrix}
A_1 & A_{12} & \cdots & A_{1N} \\
A_{21} & A_2 & \cdots & A_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
A_{N1} & A_{N2} & \cdots & A_N
\end{bmatrix}, \quad B = \text{diag}(B_1, \ldots, B_N) \]

and LMI-based decentralised controllers are proposed by using the robust control method [10, 13].
where $J. \text{Eng.}$

Substituting (8) into (7), we can obtain

$$W_i = -(QAx_i + A_i^TQ_i - QBB_i^TQ_i + \sum_{j=1, j \neq i}^{N} \mu_j H_j^TH_j).$$

We have

$$V_i \leq -\min \left\{ \frac{1}{1+\mu_j} \right\} V_i.$$  \hfill (10)

Therefore, $V_i$ exponentially converges to zero, and so does each $x_i$. This completes the proof. $\blacksquare$

Remark 1: Theorem 1 presents a decentralised controller to stabilise the large-scale system with interconnections satisfying matching conditions. Compared to the step-by-step decentralised control method proposed in [1], a significant feature of the decentralised controller (4) is that it allows each subsystem implementing its control input separately.

Remark 2: $\mu_{ij}$ are weighting coefficients. If $\mu_{ij} \rightarrow 0$, the Ricatti inequality (5) will be simplified into

$$QA_i + A_i^TQ_i - QBB_i^TQ_i < 0,$$  \hfill (11)

and the feedback gain will be infinite. This confirms with the initial idea of decentralised controller design to enlarge the diagonal blocks to cover the effects of interconnections.

In the following, we consider the interconnected system of two subsystems

$$\begin{align*}
  x_1 &= A_1x_1 + B_1u_i + A_{12}x_2 \\
  x_2 &= A_2x_2 + B_2u_i + A_{21}x_1.
\end{align*}$$  \hfill (12)

we assume that $A_{12}$ can be composed into $A_{12} = B_2H_{12}$, but this does not happen to $A_{21}$. In this setting, we design $u_i = K_ix_i$ to stabilise the interconnected system in (12).

The feedback gain matrices are designed as follows:

$$\begin{align*}
  K_i &= -\frac{1}{2}B_i^TQ_i \\
  K_i &= -\frac{1}{2}B_i^TQ_i,
\end{align*}$$  \hfill (13)

where $Q_1$ and $Q_2$ are the solutions of

$$\begin{align*}
  QA_i + A_i^TQ_i - QBB_i^TQ_i + \mu_iA_i^TQ_iA_i &< 0, \\
  QA_i + A_i^TQ_i - QBB_i^TQ_i + \mu_iH_i^TH_i &+ \frac{1}{\mu_j}I &< 0,
\end{align*}$$  \hfill (14) \hfill (15)

respectively, and $\mu_i, \mu_j$ are positive constants, $\mu_i \geq 1 + (1/\mu_j)$.

Theorem 2: The interconnected system (12) can be stabilised by the decentralised static controller designed in (13).

Proof: Consider the following Lyapunov function candidate:

$$V_2 = x_1^TQ_1x_1 + x_2^TQ_2x_2.$$  \hfill (16)

The time derivative of $V_2$ along the trajectory of (12) can be written as

$$\begin{align*}
  V_2 = \sum_{j=1, j \neq i}^{N} \mu_j H_j^TH_j + x_1^TQA_i + A_i^TQ_i - QBB_i^TQ_i + 2x_2^TQB_iH_{1j}x_j. \\
  x_2^TQB_iH_{1j}x_j \leq \frac{1}{\mu_j}x_2^TQB_iB_i^TQ_i + \mu_j x_2^TQ_iH_{1j}H_{1j}x_j.
\end{align*}$$  \hfill (17)

Substituting (8) into (7), we can obtain

$$\begin{align*}
  V_2 \leq x_1^TQA_i + A_i^TQ_i - QBB_i^TQ_i + 2x_2^TQB_iB_i^TQ_i + \sum_{j=1, j \neq i}^{N} \mu_j H_j^TH_j \\
  \quad + x_1^TQA_i + A_i^TQ_i - QBB_i^TQ_i \leq 0.
\end{align*}$$  \hfill (9)

where the second inequality is derived from (5). Define

$$\begin{align*}
  I &= \frac{1}{2}B_i^TQ_i \\
  I &= \frac{1}{2}B_i^TQ_i.
\end{align*}$$  \hfill (13)
2x_i^TQ_iB_iH_{ij}x_j \leq \frac{1}{\mu_i}x_i^TQ_iB_iB_i^TQ_i x_i + \mu_i x_i^T H_{ij}^T H_{ij}x_i \tag{18}

and
\begin{equation}
2x_i^TQ_iA_{ij}x_j \leq \mu_i x_i^T Q_i A_{ij}^T Q_i A_{ij} x_j + \frac{1}{\mu_i} x_i^T x_i . \tag{19}
\end{equation}

We can obtain
\begin{equation}
V_2 \leq x_i^T(Q_iA_i + A_i^TQ_i - Q_iB_iB_i^TQ_i) + \mu_i x_i^T Q_i A_{ij}^T Q_i x_j + x_j^T(Q_jA_j + A_j^TQ_j - Q_jB_jB_j^TQ_j) - Q_iB_iB_i^TQ_i + \mu_i H_{ij}^T H_{ij} + \frac{1}{\mu_i} x_i^T x_i \leq 0 . \tag{20}
\end{equation}

Similar to the proof of Theorem 1, we can easily know that \( V_2 \) and thus \( x_1 \) and \( x_2 \) exponentially converge to zero. \( \square \)

**Remark 3:** Note that the condition to realise decentralised control is of less conservativeness, which allows that the interconnection matrix of one subsystem does not satisfy matching condition. Also note that the decentralised controller designed in (13) is not separate, but with the step-by-step method since the determination of \( Q_i \) depends on the selection of \( Q_j \).

**Theorem 2:** It gives a relaxed sufficient condition for decentralised control of two interconnected systems. In the following, we will extend the results to the general case that the interconnected system with \( N \) subsystems, as described in (1).

We make some modification on Assumption 1 to get the following relaxed matching condition.

**Assumption 2:** There is only one subsystem \( k \) whose interconnection matrices \( A_{kj}, j = 1, \ldots, k - 1, k + 1, \ldots, N \) cannot be decomposed as \( A_{kj} = B_jH_{kj} \).

Under Assumption 2, we have the following decentralised controller:
\begin{align}
K_i &= -\frac{\mu_i}{2}B_i^TQ_i , & i \neq k \\
K_k &= -\frac{1}{2}B_i^TQ_i, 
\end{align}
\tag{21}

where \( Q_i \) and \( Q_k \) are the solutions of
\begin{equation}
Q_iA_i + A_i^TQ_i - Q_iB_iB_i^TQ_i + \sum_{j=1, j \neq i, k}^{N} \mu_j H_{ji}^T H_{ji} + \mu_k A_i^T Q_i A_{ij} < 0 , \tag{22}
\end{equation}

and
\begin{equation}
Q_kA_k + A_k^TQ_k - Q_kB_kB_k^TQ_k + \sum_{i=1, i \neq k, j}^{N} \left[ \mu_k H_{ik}^T H_{ik} + \frac{1}{\mu_k} \right] A_{ij} < 0 , \tag{23}
\end{equation}

respectively, and \( \mu_i, \mu_j, \) and \( \mu_k \) are positive constants, \( \mu_i \geq 1 + \sum_{j=1, j \neq i, k}^{N} \mu_j \).

**Theorem 3:** Suppose Assumption 2 holds. The interconnected system (1) can be stabilised by the decentralised static controller designed in (21).

**Proof:** Consider the following Lyapunov function candidate:
\begin{equation}
V_i = \sum_{j=1}^{N} x_j^TQ_j x_j . \tag{24}
\end{equation}

The time derivative of \( V_1 \) can be written as
\begin{equation}
\dot{V}_1 = \sum_{i=1}^{N} x_i^T(Q_iA_i + A_i^TQ_i - \mu_i Q_iB_iB_i^TQ_i)x_i \\
+ \sum_{i=1}^{N} x_i^TQ_iB_iH_{ij}x_j \\
+ \sum_{i=1}^{N} 2x_i^TQ_iA_{ij}x_j \\
+ x_j^T(Q_jA_j + A_j^TQ_j - Q_jB_jB_j^TQ_j)x_k . \tag{25}
\end{equation}

Similarly, we have
\begin{equation}
2x_i^TQ_iB_iH_{ij}x_j \leq \frac{1}{\mu_j} x_i^TQ_iB_iB_i^TQ_i x_i + \mu_j x_j^T H_{ij}^T H_{ij}x_j \tag{26}
\end{equation}

and
\begin{equation}
2x_i^TQ_iA_{ij}x_j \leq \mu_i x_j^T A_{ij}^T Q_i A_{ij} x_j + \frac{1}{\mu_i} x_i^T x_i . \tag{27}
\end{equation}

Substituting (26) and (27) into (25) yields
\begin{equation}
\dot{V}_i = \sum_{i=1}^{N} x_i^T(Q_iA_i + A_i^TQ_i - Q_iB_iB_i^TQ_i) + \sum_{j=1, j \neq i, k}^{N} \mu_j H_{ji}^T H_{ji} + \mu_k A_i^T Q_i A_{ij}x_i \\
+ x_j^T(Q_jA_j + A_j^TQ_j - Q_jB_jB_j^TQ_j) + \sum_{i=1, i \neq k, j}^{N} \left[ \mu_k H_{ik}^T H_{ik} + \frac{1}{\mu_k} \right] A_{ij} \leq 0 . \tag{28}
\end{equation}

The rest of the proof is similar to that of Theorems 1 and 2, which is omitted here for brevity. \( \square \)

Since controller (21) is designed step-by-step, we have the following algorithm to get the controller.

**Algorithm 1:**

1. **Step 1:** Solve the Ricatti inequality (23) to get \( Q_k \) and \( K_k = - (1/2)B_k^TQ_k \).

2. **Step 2:** Solve the Ricatti inequality (22) to get \( Q_i \) and \( K_i = - (\mu_i/2)B_i^TQ_i \).

**Remark 4:** Although Algorithm 1 is a step-by-step method, it contains only two steps. The first step is to design the controller for the subsystem \( k \) and the second step is to design controllers for the rest \( N-1 \) subsystems. The controllers for \( N-1 \) subsystem are separate, which are all dependent on the subsystem \( k \). This is mainly caused by relaxing sufficient condition from Assumption 1 to Assumption 2.

## 3 Decentralised dynamic controller design

In the previous section, we have presented decentralised static controllers. In practise, the decentralised static controller always has the disadvantage that the control input may be over a threshold value.

In this section, we are aimed to propose decentralised dynamic controller.

Consider the interconnected system of two subsystems in (12). We have the following decentralised dynamic controller:
\begin{equation}
u_i = K_i x_i - H_i y_i \\
v = (A_i + B_i K_i)^2 + A_i x_i, \tag{29}
u_i = K_i x_i,
\end{equation}
where $K_1$ and $K_2$ are matrices such that $A_1 + B_1K_1$ and $A_2 + B_2K_2$ are Hurwitzes, $v$ is the designed observer for the first subsystem to observe the state of $x_2$.

**Theorem 4:** The interconnected system (12) can be stabilised by the decentralised dynamic controller designed in (29).

**Proof:** Substituting (29) into (12) and define $\tilde{v} = x_2 - v$, we can get the closed-loop system

$$
\begin{align*}
\dot{x}_1 &= (A_1 + B_1K_1)x_1 + A_3\tilde{v}, \\
\dot{\tilde{v}} &= (A_1 + B_1K_1)\tilde{v}, \\
\dot{x}_2 &= (A_1 + B_1K_1)x_1 + A_{13}x_1.
\end{align*}
$$

Since $A_1 + B_1K_1$ is Hurwitz, we have that $\tilde{v}$ asymptotically converges to zero, which combining with the fact that $A_1 + B_1K_1$ is Hurwitz implies that $\tilde{x}_1$ asymptotically converges to zero, and thus $x_2$ also asymptotically converges to zero. This completes the proof.

**Remark 5:** In fact, by introducing an extra state vector $\tilde{v}$, the closed-loop system can be rewritten as

$$
x = A_{cl}x,
$$

where $x = [\tilde{v}, x_1^T, x_2^T]^T$ and

$$
A_{cl} = \begin{bmatrix}
A_1 + B_1K_1 \\
H_{12} \\
A_{13} + B_1K_2
\end{bmatrix}.
$$

Since $A_{cl}$ is a lower triangular matrix, and the diagonal elements are all Hurwitz, it is obvious that $A_{cl}$ is Hurwitz.

**Remark 6:** Note that the observer $v$ designed in (29) is to estimate $x_2$. The interconnected system (12) can be understood as follows. Here, $x_1$ is viewed as state and $x_2$ is external disturbance with known dynamics. In this way, the decentralised control problem of two interconnected systems is turned into a disturbance rejection problem [15].

### 4 Simulation

**Example 1:** For the interconnected system (1) with four subsystems, let the matrices be as follows:

$$
A_1 = A_3^T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\
B_1 = B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
A_2 = A_4^T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\
B_2 = B_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
$$

$$
A_{12} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & −1 \end{bmatrix}, \\
A_{31} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
A_{14} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
A_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
$$

and the rest interconnected matrices are zero. Note that the interconnected matrices $A_{12}$ and $A_{31}$ do not in the column space of $B_1$. Such interconnected system satisfies Assumption 2.

Choose $\mu_{ij} = 1$. Solving the Ricatti inequality (23), we can get

$$
Q_1 = \begin{bmatrix} 4.0487 & 1.7321 \\ 1.7321 & 2.3375 \end{bmatrix}, \\
K_1 = \begin{bmatrix} −0.8660 & −1.1688 \end{bmatrix}.
$$

Solving the Ricatti inequalities (22), we can get

$$
$$

and

$$
K_2 = \begin{bmatrix} −3.0764 & −6.2649 & −7.5045 \end{bmatrix}.
$$

The time trajectories of the states of the interconnected system are given in Fig. 1, demonstrating that the decentralised controller can stabilise the interconnected system.

**Example 2:** For the two interconnected system (12), let

$$
A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\
B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
$$

$$
A_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & −1 \end{bmatrix}, \\
A_{31} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}, \\
A_{32} = \begin{bmatrix} 3 & 0 \\ −2 & 3 \end{bmatrix},
$$

Choose $\mu_{i1} = 1$. Solving the Ricatti inequality (15), we can get

$$
$$

and

$$
K_2 = \begin{bmatrix} −0.8660 & −1.9629 & −1.6471 \end{bmatrix}.
$$

Solving the Ricatti inequality (14), we can get

$$
Q_1 = \begin{bmatrix} 198.0547 & 20.9800 \\ 20.9800 & 43.2637 \end{bmatrix},
$$

and

$$
K_1 = \begin{bmatrix} −20.9800 & −43.2637 \end{bmatrix}.
$$

The state trajectories of two interconnected systems are given in Fig. 2, demonstrating that the decentralised static controller (13) can stabilise the interconnected system (12).

Next, we use the decentralised dynamic controller to stabilise the two interconnected systems (12). Choose...
\[ K_1 = [-0.5000 \quad -0.8660] \quad \text{and} \quad K_2 = [-0.5000 \quad -1.2071 \quad -1.2071] \]
such that \( A_1 + B_1K_1 \) and \( A_2 + B_2K_2 \) are Hurwitzes. Note that the control gain matrix in the dynamic controller is much smaller than that of the static controller. The trajectories of the states and the observer are given in Fig. 3, demonstrating that the decentralised dynamic controller (29) can stabilise the interconnected system (12).
5 Conclusion

This paper works on the decentralised controller design of interconnected systems. Relaxed sufficient condition for the existence of decentralised controller is presented, based on which decentralised static controller is proposed. Moreover, the decentralised dynamic controller is also designed for interconnected systems with two subsystems. The controllers designed in this paper is either totally separate or in two steps and thus can be easily implemented in practise.

6 References