Multi-adjoint intuitionistic fuzzy rough sets

Meishe Liang¹², Jusheng Mi¹, Tao Feng³, Tianna Zhao¹
¹College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang 050024, People’s Republic of China
²Department of Scientific Development and School-Business Cooperation, Shijiazhuang University of Applied Technology, Shijiazhuang 050081, People’s Republic of China
³College of Science, Hebei University of Science and Technology, Shijiazhuang 050018, People’s Republic of China
E-mail: mijsh@263.net

Abstract: The combination of fuzzy information systems (IFSs) and multi-adjoint theory has become a hot issue in the study and applications of artificial intelligence. An intuitionistic fuzzy set has more flexible and practical ability to represent information and is better in dealing with ambiguity and uncertainty when compared with the fuzzy set. Multi-adjoint intuitionistic fuzzy rough sets are constructed by using adjoint triples under intuitionistic fuzzy IS. For this purpose, the authors propose intuitionistic fuzzy indiscernibility relation and multi-adjoint approximation operators. The basic results in the multi-adjoint fuzzy rough set model are generalised to multi-adjoint intuitionistic fuzzy rough set model. The analogous results are also verified. After that, a novel approach of attribute reduction is proposed. First, a kind of approximate reduction to keep the dependence of the positive region to a degree α is formulated. Second, they propose a heuristic algorithm to compute the attribute reduction. At last, they employ an example to describe the processing of the algorithm.

1 Introduction

The rough set was originated by Pawlak [1]. As an important generalisation of the classical set theory, it has been a powerful mathematical tool for modelling and handing incomplete information in relational databases, and widely used in machine learning, data mining etc. The early works of the rough set theory were mainly used for processing the discrete data sets only, where each object took a unique discrete value for each attribute. According to unique discrete value, each attribute can induce an equivalence relation in such a data set. However, the equivalence relation is too restrictive that cannot be used for many applications. To tackle more real-world application problem, many scholars generalised the equivalence relation with dominance relation [2], similarity relation [3], fuzzy relation [4, 5] and tolerance relation [6], during the past 30 years.

Adjoint triple is composed of t-norm and residuated implications which satisfy the adjointness property [7]. Combined with concept lattice theory, there are many novel models such as multi-adjoint concept lattices [8], multi-adjoint t-concept lattices [9], multi-adjoint property-oriented concept lattices and object-oriented concept lattices [10], dual multi-adjoint concept lattices [11] etc. These models greatly enriched the theory of concept lattice. Combined with the rough set theory, Comelis et al. [12] introduced multi-adjoint fuzzy rough set. Those models increase the flexibility of the approximation operators in the considered data set.

The fuzzy set was proposed in earlier works by Zadeh [13]. In fuzzy set theory, a membership function was used to characterise the imprecision, uncertainty and approximation by attributing a degree to which a certain object belongs to a set [14]. Moreover now, the fuzzy set theory has revolutionised many research fields. On the basis of this theory, several extensions of the fuzzy set for various practical applications have been proposed. As its consequence, intuitionistic fuzzy set (IFS) was proposed by Atanassov [15, 16] in the 1980s. By considering the membership, non-membership, and hesitation of the three aspects of information [17], this model was more flexible and practical in dealing with ambiguity and uncertainty than Zadeh’s. After that, Takeuti and Titani [18] defined intuitionistic fuzzy (IF) logic and IF set by consideration the propositions valued into the range [0, 1]. In recent years, the research on the combination of IFS and the rough set has received widespread attention [19–22]. After that, a new dominance relation was introduced by Zhang and Chen [23], and they then proposed generalised dominance-based IF rough set model. Considering the covering IF approximation space, Xue et al. [24] defined a new covering rough IF set model. By means of a similarity measure, Tiwari et al. [25] proposed a quick algorithm to calculate attribute reduct in tolerance-based IF rough set theory. Huang et al. [26] developed a new multigranulation rough set in IF information system (IFS), called IF multigranulation rough set. During the past few years, many types of rough approximation operators [27, 28] and measures [29–31] of intuitionistic fuzzy sets have been proposed. Inspired by Hamming, Euclidean and Hausdorff metrics, Singh and Gang [32] presented a family of distance measures based on in type-2 IFS. Huang et al. [33] extended the classical decision theory rough set (DTRS), in which a type of inclusion measure between two IFSs was introduced. A general framework was proposed by Zhou and Wu [34], in which both constructive and axiomatic approaches were used for the study of relation-based intuitionistic fuzzy rough approximation operators.

Although some researchers have presented multi-adjoint approaches in fuzzy IS, but none of them have considered the model in intuitionistic fuzzy IS (IFIS). Thus, it is very necessary to do the research based on multi-adjoint theory in IFIS. In this paper, by using the operators formed by an adjoint triple, we define the multi-adjoint IF rough set model, which can increase the flexibility of handling uncertainty in a very effective manner by combining two effective tools, i.e. adjoint triple theory and IF set theory. This model has also retained the main features of adjoint triples, that is, explicit preferences among the objects may be represented by using different adjoint triples. On the basis of the above-mentioned advantages, we have defined a degree of intuitionistic fuzzy dependence, which is employed to acquire approximate reducts with respect to decision attribute. According to the significance of attributes, we propose a heuristic algorithm to compute the attribute reducts. After that, an example is employed to illustrate the processing of the algorithm.
The rest of this work is structured as follows. In Section 2, we briefly introduce the preliminary definitions, i.e. rough sets and multi-adjoint fuzzy rough sets in this paper. Next, in Section 3, multi-adjoint IF rough set is defined and the main propositions of the model are given in Section 4. In Section 5, we define a new dependence of the positive region and then discuss the ways of attribute selection in this framework. We conclude this entire paper in Section 6.

2 Preliminaries

2.1 Rough set

In classical rough set theory [1], IS is also called information table, and always defined as $IS = (U, A)$, where the objects set $U = \{x_1, x_2, ..., x_n\}$ and the attributes set $A = \{a_1, a_2, ..., a_m\}$ are non-empty, finite sets. For each attribute $a \in A$, the relation functions between $U$ and $A$ are defined as $f_a: U \times A \rightarrow V_a$ where $V_a$ consists all values of $a$ over $U$. With each non-empty subset $B \subseteq A$, we associate a binary relation $R_B$, which is called the $B$-indiscernibility relation (or the equivalence relation), and defined by

$$R_B = \{(x, y) \in U \times U: f_a(x) = f_a(y)\}.$$  

According to $R_B$, we can get a partition on $U$. That is a family of disjoint subsets $\{[x]_B: x \in U\}$. $[x]_B$ is named as an equivalence class of $x$ according to $R_B$ and defined by

$$[x]_B = \{y \in U: (x, y) \in R_B\}.$$  

Given $X \subseteq U$, a pair of upper approximation and lower approximation operators with respect to $B$ are defined, respectively, by

$$\overline{R_B}(X) = \{x \in U: [x]_B \cap X \neq \emptyset\};$$

$$\underline{R_B}(X) = \{x \in U: [x]_B \subseteq X\}.$$  

If $\overline{R_B}(X) = R_B(X)$, $X$ is a set that can be defined accurately. If not, the pair $(\overline{R_B}(X), \underline{R_B}(X))$ is known as a rough set of $X$.

A decision IS $DIS = (U, A \cup D)$, where $D = \{d\}$ and $d \notin A$ are called a decision attribute. According to $d$-indiscernibility relation, each $x \in U$ belongs to a decision equivalence class. For every non-empty subset $B \subseteq A$, the positive region and the positive region dependency with respect to $B$ are defined, respectively, as

$$POS_B(D) = \bigcup_{x \in U} R_B([x]_B);$$

$$\gamma_B = \frac{|POS_B(D)|}{|U|}.$$  

If $\gamma_B = 1$, then we say that the partition of $U$ by $A$ is finer than by $D$, and $DIS = (U, A \cup D)$ is a consistent DIS. If a subset $B$ of $A$ satisfies $POS_B(D) = POS_A(D)$, and there exists no proper subset $B'$ of $B$ satisfies $POS_{B'}(D) = POS_A(D)$, then $B$ is called a decision reduct.

2.2 Multi-adjoint fuzzy rough sets

As well known, the adjective triple theory provides a new point of view to acquire knowledge from the information table such as in incomplete IS and imprecise IS [9]. The most important feature is that the adjoint triples are flexible to increase the representing preferences among the objects in the considered framework. This section introduces the main definition of multi-adjoint fuzzy rough sets in brief.

**Definition 1:** Given three posets $(P_1, \preceq_1)$, $(P_2, \preceq_2)$ and $(P_3, \preceq_3)$, the pair $(\&_1, \rightarrow_1, \leftrightarrow_1)$ is called an adjoint triple with respect to $P_1$, $P_2$, $P_3$, if the three mappings and: $P_1 \times P_2 \rightarrow P_3,$

$\preceq: P_1 \times P_2 \rightarrow P_3,$

$\nabla: P_1 \times P_2 \rightarrow P_3$,

satisfy

(1) $x \preceq z$ iff $x \& y \preceq z$ and $y \preceq z \nabla x$, for all $x \in P_1$, $y \in P_2$ and $z \in P_3$;

(2) $\nabla$ is a $t$-norm and order preserving on both arguments; and

(3) $\preceq$, $\nabla$ are order preserving on the first argument and order preserving on the second argument [7].

There are three special examples of adjoint triples:

**Gödel adjoint triple**

$$x \&_G y = \min \{x, y\}, \nabla \nabla G x = \{1 \text{ if } x \leq z, \text{ otherwise}\};$$

**Product adjoint triple**

$$x \&_P y = xy, \nabla \nabla P x = \min \{1, z/x\};$$

**Łukasiewicz adjoint triple**

$$x \&_L y = \max \{0, x + y - 1\}, \nabla \nabla L x = \min \{1, 1 - x + z\}.$$  

The above adjoint triples are formed by $t$-norms and two residuated implications, which can be considered to be degenerate examples of general adjoint triples, and in these three adjoint triples $\nabla = \nabla G$, $\nabla = \nabla P$, $\nabla = \nabla L$ on account of $\&_G$, $\&_P$, $\&_L$ to be commutative.

**Lemma 1:** Let $(P_1, \preceq_1)$ be a poset, $(U_2, \preceq_2)$ and $(U_3, \preceq_3)$ be two complete lattices. If the pair $(\&_2, \rightarrow_2, \leftrightarrow_2)$ is an adjoint triple with respect to $P_1$, $P_2$, $P_3$, then

(1) $x \&_2 \{\sup \{y \mid i \in J\}\} = \{\sup \{x \&_2 y \mid i \in J\}\};$

(2) $\inf \{z \mid j \in I\} \rightarrow_2 x = \inf \{\nabla \nabla_2 x \mid j \in I\}.$

where $I$ and $J$ are two index sets and $x \in P_1$, $y \in P_2$, $\{z \mid j \in J\} \in P_3$.

Throughout this section, we fix a fuzzy decision information table $DIS = (U, A \cup D)$, a complete lattice $(L, \leq)$ and a poset $(P, \leq)$ together. Assume the poset $(P, \leq)$ has the top element $T_P$. Let $B \subseteq A$, Comelis et al. [12] first introduced a fuzzy generalisation $B$-indiscernibility relation. That is a $P$-fuzzy tolerance relation: $R_B: U \times U \rightarrow P$, and $R_B$ satisfies reflexivity property, $\forall x \in U$, $R_B(x, x) = T_P$; and symmetric property, $\forall x, y \in U$, $R_B(x, y) = R_B(y, x)$.

**Definition 2:** Let $DIS = (U, A \cup D)$ be a fuzzy decision information table. $R_B: U \times U \rightarrow P$ is $P$-fuzzy tolerance relations, for every $a \in A$. Given a subset $B$ of $A$, fuzzy $B$-indiscernibility relation is also a $P$-fuzzy tolerance relations, defined, $\forall x, y \in U$, as

$$R_B(x, y) = \ominus (\phi_B^a \ominus a),$$

in which $\ominus : P^n \rightarrow P$ is an aggregation operator on each argument, and mapping $\phi_B^a: A \rightarrow P$ is defined for each attribute $a \in A$ as

$$\phi_B^a = \begin{cases} R_B(x, y) & \text{if } a \in B \\ T_P & \text{otherwise} \end{cases}.$$  

Additionally, $\ominus (T_P, ..., T_P) = T_P$, and if $\perp_P$ is a bottom element of the poset $(P, \preceq)$, then $\ominus (\perp_P, ..., \perp_P) = \perp_P$. All fuzzy sets in $U$ are denoted by $F(U)$. Considering the context $(U, U, R_B, \tau)$, $\tau$ maps all pairs of elements in $U \times U$ to particular adjoint triples in the context above. $\forall g \in F(U)$, the multi-adjoint upper and lower approximation operators denoted by $g^+\tau$ and $g^{-\tau}$, respectively, are defined as follows:

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To improve readability and simplify the writing, we use &\in\cap, \setminus, \cup to take the place of \&\&_{x,y}, \setminus_{x,y}, \cup_{x,y} in the following content. The pair \((g^{i,N}, g^{i,T})\) is called a multi-adjoint fuzzy rough set.

Using the multi-adjoint lower approximation operator, the positive region of \(D\) with respect to \((w.r.t.) B\) is defined as, for each \(y \in U\)

\[
\text{POS}_B(D)(y) = \left\{ \left( \bigcup_{x \in U} (R_B(x))^i \right) \cap (R_B(x))^T \right\} \\
= \sup_{x \in U} \left\{ \left( R_B(x) \cap \mu_{\cap} \right) \cap \left( R_B(x) \cap \mu_{\cap} \right) \right\}
\]

(10)

where \(R_B: U \times U \rightarrow P\) is a \(P\)-fuzzy relation. If the relationship of \(d\) is crisp, then \(R_B\) partitions object set \(U\) into a family of decision classes. Therefore, there are only two regions for \(R_B(x, y)\).

\[
\text{POS}_B(D)(y) = \inf_{x \in U} \left\{ (R_B(x, y))^i \right\}
\]

(11)

3 Multi-adjoint IFSs

3.1 Intuitionistic fuzzy sets

In the process of human cognition, a large number of concepts have the characteristics of fuzziness. As a generalisation of fuzzy set, intuitionistic fuzzy relations have more objective and exquisite description of fuzziness in the real world. In this section, we will briefly review the conceptions of the IF sets and the rules of operators, which will be used for the following sections of our work.

Definition 3: Let \(U\) be a non-empty and finite universe of discourse. An IF set \(A\) in \(U\) has the form \(A = (\mu_A(x), \nu_A(x)): x \in U\), where the mappings \(\mu_A: U \rightarrow [0, 1]\) and \(\nu_A: U \rightarrow [0, 1]\) denote the degree of membership and non-membership of each element \(x \in U\) to \(A\), respectively, and \(0 \leq \mu_A(x) + \nu_A(x) \leq 1\) for each \(x \in U\). We denote \(1 - \mu_A(x) - \nu_A(x)\) as the hesitancy degree of \(x\) to \(A\) [15].

All IF sets in \(U\) are denoted by \(\mathcal{IF}(U), \forall x \in U\).

Definition 4: Let \(U\) be a non-empty and finite universe of discourse and

\[
\begin{align*}
(1) & \ A \subseteq B \iff \forall x \in U, \mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \leq \nu_B(x); \\
(2) & \ A = B \iff A \subseteq B \text{ and } B \subseteq A \\
(3) & \ \sim A = \{ (\nu_A(x), \mu_A(x)) : x \in U \} \\
(4) & \ A \cap B = \{ (\min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))) : x \in U \} \\
(5) & \ A \cup B = \{ (\max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))) : x \in U \}
\end{align*}
\]

(12)

According to the fuzzy IS, we similarly introduce the notion of IFIS.

Definition 5: A tuple IFIS \(= (U, A, V, f)\) is called an IFIS, where \(U = (x_1, x_2, \ldots, x_n)\) and \(A = \{a_1, a_2, \ldots, a_m\}\) are finite, non-empty sets of objects and attributes, respectively. \(V\) is the range of the attributes and defines in \([0, 1] \times [0, 1]\). Mapping \(f: U \times A \rightarrow V\) is defined as \(f(x, a) = (\mu_{aA}(x), \nu_{aA}(x))\), where \(0 \leq \mu_{aA}(x) + \nu_{aA}(x) \leq 1\), \(\forall a \in A\).

3.2 Construction of multi-adjoint IF rough approximation operators

First, we introduce the construction of intuitionistic fuzzy indiscernibility relation between two objects \((w.r.t.) A\) attribute subset \(B \subseteq A\). As we know, fuzzy IS can determine a two-dimensional (2D) data table, and IFIS also determines a 2D data table. Unlike the former, each element is composed of two parts. So we have to generalise Definition 2.

Definition 6: Let \(IFIS = (U, A, V, f)\) be an IFIS. \(\forall U \in A\), we define a mapping: \(IR: U \times U \rightarrow [0, 1] \times [0, 1]\) to characterise the intuitionistic fuzzy indiscernibility relation between two objects according to the attribute \(A\), that is, \(\forall x, y \in U\)

\[
IR(x, y) = (\mu_{IR}(x, y), \nu_{IR}(x, y)), \quad 0 \leq \mu_{IR}(x, y) + \nu_{IR}(x, y) \leq 1.
\]

Moreover, \(\mu_{IR}(x, y), \nu_{IR}(x, y)\) are \(P\)-fuzzy positive and negative relationships \((w.r.t.) A\) objects \(x, y\) in IFIS, which satisfy the reflexivity property, \(\forall x \in U, \mu_{IR}(x, x) = 1, \nu_{IR}(x, x) = 0\); moreover, the symmetric property, \(\mu_{IR}(x, y) = \mu_{IR}(y, x), \nu_{IR}(x, y) = \nu_{IR}(y, x), \forall x, y \in U\).

Definition 7: Let \(IFIS = (U, A, V, f)\) be an IFIS. For \(B \subseteq A\), \(\forall x, y \in U\), the \(B\)-intuitionistic fuzzy indiscernibility relation is defined as

\[
IR_B(x, y) = (\mu_{IR}(x, y), \nu_{IR}(x, y))
\]

(13)

\[
= (\min(\mu_B^\gamma(a)), \dots, \mu_B^\gamma(a)), \quad (\max(\nu_B^\gamma(a)), \dots, \nu_B^\gamma(a))
\]

(12)

in which \(\mu_{IR}(x, y), \nu_{IR}(x, y)\) are also \(P\)-fuzzy tolerance relations and \(\ominus: \mathbb{P}^m \rightarrow \mathbb{P}\) is a monotonic aggregation operator, \(\forall U \in A\)

\[
\mu_B^\gamma(a) = \begin{cases} 
\mu_{IR}(x, y), & a \in B \\
1, & \text{otherwise}
\end{cases}
\]

(14)

From Definition 7, it is easy to prove that if \(B \subseteq B \subseteq A\), we have \(IR_B \subseteq IR_B\). Let

\[
\mu_{IR_B}(x, y) = \inf\{ (\mu_{IR}(x, y), \nu_{IR}(x, y)) : y \in U \}
\]

(13)

\[
\nu_{IR_B}(x, y) = \sup\{ (1 - \mu_{IR}(x, y)) \cup \nu_{IR}(x, y) : y \in U \}
\]

(12)

\[
\nu_{IR_B}(x, y) = \sup\{ (\mu_{IR}(x, y) \cap \nu_{IR}(x, y)) : y \in U \}
\]

(14)

\[
\nu_{IR_B}(x, y) = \inf\{ (\nu_{IR}(x, y) \cup (1 - \mu_{IR}(x, y))) : y \in U \}
\]

According to Definition 2, \(\forall x \in U\), \(0 \leq \nu_{IR_B}(x, y) \leq 1\). Therefore, \(0 \leq \nu_{IR_B}(x, y) \leq \mu_{IR_B}(x, y) \leq 1\). Combined with the notion of adjoint triple, \(\forall A \in \mathcal{IF}(U)\) the intuitionistic fuzzy upper and the lower approximations of \(A\) can be acquired.

Definition 8: Let \(IFIS = (U, A, V, f), \forall A \in \mathcal{IF}(U)\), the lower and the upper approximations of \(A\) \((w.r.t.) A\) attribute subset \(B\) are defined by \(IR_B(A)\) and \(IR_B(A)\), respectively, in which

\[
IR_B(A) = \{ (\mu_{IR_B}(x), \nu_{IR_B}(x)) : x \in U \}
\]

(13)

\[
IR_B(A) = \{ (\mu_{IR_B}(x), \nu_{IR_B}(x)) : x \in U \}
\]

(14)
In this section, we study the main properties of the multi-adjoint intuitionistic fuzzy rough set.

**Proposition 1:** Let \( \text{IFIS} = (U, A, V, f) \), \( \forall A \in \mathcal{IF}(U) \), \( \overline{\text{IF}}(A) \subseteq \text{IF}(A) \).

**Proof:** In \( \text{IFIS} = (U, A, V, f) \), \( \forall x \in U \)

\[ u'_{\text{IR}}(A) = \inf\{\mu_A(y) \land \mu_B(x); y\in U\} \leq \mu_A(x) \land 1 = \mu_A(x); \]

\[ \mu_A(A) = \{0.60, 0.30\}, \{0.60, 0.30\}, \{0.58, 0.30\}, \{0.53, 0.30\}, \{0.43, 0.30\}, \{0.40, 0.30\}, \{0.40, 0.30\}\. 

From Definition 8, an important property of the adjoint triples is that one can associate different pairs of adjoint triples among the objects. For example, in Example 1, we assume that the values w.r.t. the objects \( x_1, x_2, x_6 \) are more trustworthy than the other objects. Therefore, we associate \( x_1, x_2, x_6 \) with the product adjoint triple replace the Gödel adjoint triple, since as we know the Gödel implication has more influence on \( \mu_{IR}(A) \) and \( \tau_{IR}(A) \).

**Example 2:** Consider the context \( (U, U, \text{IR}_A, r) \) from Example 1. If \( x_i, x_j \in \{x_1, x_2, x_6\} \), let \( r_{x_i} = \tau_r \); otherwise, \( r_{x_i} = \tau_G \). By Definition 8, we have

\[ \text{IR}_A(A) = \{0.25, 0.56\}, \{0.20, 0.60\}, \{0.20, 0.70\}, \{0.20, 0.70\}, \{0.25, 0.65\}, \{0.20, 0.70\}\; \{0.60, 0.30\}, \{0.60, 0.30\}, \{0.58, 0.30\}, \{0.53, 0.30\}, \{0.40, 0.40\}, \{0.40, 0.40\}, \{0.40, 0.30\}\. 

The results show that the lower approximation is greater than the previous one, the upper smaller than the previous one in this particular example. In fact, we associate different adjoint triples among the objects which provide different preferences among objects.

**4 Properties**

In this section, we study the main properties of the multi-adjoint intuitionistic fuzzy rough set.

**Proposition 1:** Let \( \text{IFIS} = (U, A, V, f) \), \( \forall A \in \mathcal{IF}(U) \), \( \text{IR}_A(A) \subseteq \text{IF}(A) \).

**Table 1** IFIS = (U, A, V, f)

<table>
<thead>
<tr>
<th></th>
<th>X1</th>
<th>X2</th>
<th>X3</th>
<th>X4</th>
<th>X5</th>
<th>X6</th>
<th>X7</th>
</tr>
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<tbody>
<tr>
<td>a1</td>
<td>(0.90, 0.05)</td>
<td>(0.90, 0.05)</td>
<td>(0.10, 0.70)</td>
<td>(0.00, 0.90)</td>
<td>(0.10, 0.80)</td>
<td>(0.30, 0.60)</td>
<td>(0.00, 0.90)</td>
</tr>
<tr>
<td>a2</td>
<td>(0.70, 0.20)</td>
<td>(0.20, 0.60)</td>
<td>(0.10, 0.80)</td>
<td>(0.90, 0.10)</td>
<td>(0.10, 0.90)</td>
<td>(0.20, 0.70)</td>
<td>(0.10, 0.80)</td>
</tr>
<tr>
<td>a3</td>
<td>(0.20, 0.80)</td>
<td>(0.20, 0.50)</td>
<td>(0.10, 0.80)</td>
<td>(0.90, 0.10)</td>
<td>(1.00, 0.00)</td>
<td>(0.90, 0.00)</td>
<td>(0.90, 0.10)</td>
</tr>
<tr>
<td>a4</td>
<td>(0.70, 0.20)</td>
<td>(0.10, 0.80)</td>
<td>(0.90, 0.05)</td>
<td>(0.80, 0.10)</td>
<td>(0.80, 0.05)</td>
<td>(0.10, 0.90)</td>
<td>(0.20, 0.60)</td>
</tr>
</tbody>
</table>

**Table 2** IF indiscernibility relation IR_A

<table>
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<th>X1</th>
<th>X2</th>
<th>X3</th>
<th>X4</th>
<th>X5</th>
<th>X6</th>
<th>X7</th>
</tr>
</thead>
<tbody>
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<td>x1</td>
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<td>( (0.73, 0.25) )</td>
<td>( (0.58, 0.35) )</td>
<td>( (0.53, 0.44) )</td>
<td>( (0.43, 0.56) )</td>
<td>( (0.40, 0.59) )</td>
<td>( (0.33, 0.64) )</td>
</tr>
<tr>
<td>x2</td>
<td>-</td>
<td>( (1.00, 0.00) )</td>
<td>( (0.55, 0.40) )</td>
<td>( (0.25, 0.61) )</td>
<td>( (0.40, 0.51) )</td>
<td>( (0.68, 0.26) )</td>
<td>( (0.55, 0.36) )</td>
</tr>
<tr>
<td>x3</td>
<td>-</td>
<td>-</td>
<td>( (1.00, 0.00) )</td>
<td>( (0.55, 0.39) )</td>
<td>( (0.75, 0.23) )</td>
<td>( (0.53, 0.45) )</td>
<td>( (0.60, 0.34) )</td>
</tr>
<tr>
<td>x4</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>( (1.00, 0.00) )</td>
<td>( (0.75, 0.25) )</td>
<td>( (0.58, 0.40) )</td>
<td>( (0.65, 0.30) )</td>
</tr>
<tr>
<td>x5</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>( (1.00, 0.00) )</td>
<td>( (0.73, 0.25) )</td>
<td>( (0.80, 0.19) )</td>
</tr>
<tr>
<td>x6</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>( (1.00, 0.00) )</td>
<td>( (0.88, 0.12) )</td>
</tr>
<tr>
<td>x7</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>( (1.00, 0.00) )</td>
</tr>
</tbody>
</table>
From the property (1) in Definition 4, \( IR_{A} \) and symmetric properties. Let \( IFS = (U, A, V, f) \). For each \( B \subseteq A \), \( B \)-intuitionistic fuzzy indiscernibility relation \( IR_B \) satisfies reflexivity and symmetric properties.

Proof: It directly follows from the definition of \( IR_B \) in Definition 7.

Proposition 3: Let \( IFS = (U, A, V, f) \). \( \forall A, C \in \mathcal{F}(U) \), the following properties hold

1. \( IR_{A \cap C} = IR_A \cap IR_C \)
2. \( IR_{A \cap C} \subseteq IR_A \cap IR_C \)
3. \( IR_{A \cap C} \subseteq IR_C \cap IR_A \)
4. \( IR_{A \cup C} \supseteq IR_A \cup IR_C \)
5. \( IR_U = U \), \( IR_{\emptyset} = \emptyset \)
6. \( IR_{\emptyset}(x) = \emptyset \), \( IR_{\emptyset}(x) = \emptyset \)

Proof: For (1), \( \forall x \in U \)

\[
\mu_{IR_{A \cap C}}(x) = \sup\{(1 - \nu_B(x, y)\&_{\mu_A} \nu_A(y); y \in U) \}
\]

\[
\geq (1 - \nu_B(x, x))\&_{\mu_A} \nu_A(x)
\]

\[
= 1\&_{\mu_A} \nu_A(x)
\]

\[
= \nu_A(x).
\]

Assume that \( 0 \leq \mu_{IR_{A \cap C}}(x) + \nu_{IR_{A \cap C}}(x) \leq 1 \), it holds that \( \mu_{IR_{A \cap C}}(x) + \nu_{IR_{A \cap C}}(x) \leq \mu_A(x) \), and \( \nu_{IR_{A \cap C}}(x) = \nu_{IR_{A \cap C}}(x) \geq \nu_A(x) \); otherwise

\[
\mu_{IR_{A \cap C}}(x) = \frac{\nu_{IR_{A \cap C}}(x)}{\mu_{IR_{A \cap C}}(x) + \nu_{IR_{A \cap C}}(x)}
\]

\[
\leq \frac{\mu_A(x)}{\mu_{IR_{A \cap C}}(x) + \nu_{IR_{A \cap C}}(x)}
\]

\[
\leq \frac{\nu_A(x)}{\mu_{IR_{A \cap C}}(x) + \nu_{IR_{A \cap C}}(x)}
\]

\[
\leq \nu_A(x)
\]

\[
\geq \nu_A(x).
\]

\( \square \) From the property (1) in Definition 4, \( IR_B \) \( \subseteq A \); similarly, we have \( A \subseteq IR_B \), thus, \( IR_B \subseteq A \subseteq IR_B \).

Statement (3) is obvious. Using \( A \cap C \subseteq A, C \) and the result of Statement (1), it is easy to know that

\[
\mu_{IR_{A \cap C}}(x) \leq \mu_{IR_A}(x),
\]

\[
\mu_{IR_{A \cap C}}(x) \leq \mu_{IR_C}(x);
\]

\[
\nu_{IR_{A \cap C}}(x) \geq \nu_{IR_A}(x),
\]

\[
\nu_{IR_{A \cap C}}(x) \geq \nu_{IR_C}(x)
\]

From Definition 4 (1) and the property of infimum, we have

\( IR_B(A \cap C) \subseteq IR_B(A) \cap IR_B(C) \).

Similarly, we can prove (2) and (4).

For Statement (5), since \( A \subseteq C \Rightarrow \mu_A(x) \leq \mu_C(x) \) and \( \nu_A(x) \geq \nu_C(x), \forall x \in U \) we have

\[
\mu_{IR_{A \cap C}}(x) = \sup\{\mu_B(x, y)\&_{\mu_A} \nu_A(y); y \in U) \}
\]

\[
\leq \sup\{\mu_B(x, y)\&_{\mu_A} \nu_A(y); y \in U) \}
\]

\[
= \mu_{IR_C}(x),
\]

\[
\nu_{IR_{A \cap C}}(x) = \inf\{\nu_B(x, y) \&_{\nu_A} \mu_A(y); y \in U) \}
\]

\[
\geq \inf\{\nu_B(x, y) \&_{\nu_A} \mu_A(y); y \in U) \}
\]

\[
= \nu_{IR_A}(x).
\]

From Proposition 1 and Definition 4 (1), \( IR_B(A) \subseteq IR_B(C) \) is verified. \( IR_B(A) \subseteq IR_B(C) \) is obtained analogously.

For Statement (6), since \( IFS \) degenerate into fuzzy total set \( U \), we have \( \mu_C(x) = 1 \) and \( \nu_C(x) = 0 \), for all \( x \in U \). From Definition 7 and Proposition 1, we have

\[
1 \geq \mu_{IR_U}(x) = \sup\{\mu_B(x, y)\&_{\mu_A} \nu_A(y); y \in U) \}
\]

\[
0 \leq \nu_{IR_U}(x) = \inf\{\nu_B(x, y) \&_{\nu_A} \mu_A(y); y \in U) \}
\]

\[
\leq \nu_C(x) = 0;
\]

\[
1 \geq \mu_{IR_U}(x) = \inf\{\mu_B(x) \&_{\mu_A} \nu_A(y); y \in U) \}
\]

\[
= \inf\{\mu_B(x) \&_{\mu_A} \nu_A(y); y \in U) \}
\]

\[
= \inf\{1 \&_{\mu_A} \nu_A(y), y \in U) \}
\]

\[
\geq 1;
\]

\[
0 \geq \nu_{IR_U}(x) = \sup\{1 - \nu_B(x, y)\&_{\nu_A} \mu_A(y); y \in X) \}
\]

\[
= \sup\{1 - \nu_B(x, y)\&_{\nu_A} \mu_A(y); y \in X) \}
\]

That is \( IR_B(U) = U \), \( IR_B(U) = U \).

Statement (7) follows similarly as Statement (6).

Proposition 4: \( IR_B \), \( IR_B \) are two intuitionistic fuzzy indiscernibility relation generated by \( B_1 \) and \( B_2 \), respectively. For any \( A \in \mathcal{F}(U) \), if \( B_1 \subseteq B_2 \subseteq A \), then \( IR_B(A) \subseteq IR_B(A) \) and \( IR_B(A) \subseteq IR_B(B_2) \).

Proof: Let \( IFS = (U, A, V, f) \). Since \( B_1 \subseteq B_2 \), then \( IR_B \subseteq IR_B \), we get that \( \mu_B(x, y) \leq \mu_B(x, y) \) and \( \nu_B(x, y) \leq \nu_B(x, y) \), for all \( x, y \in U \). By Definitions 1 and 8, \( \forall x, y \in U \)

\[
\mu_{IR_{B_1 \cup B_2}}(x) = \sup\{\mu_B(x, y)\&_{\mu_A} \mu_B(y); y \in U) \}
\]

\[
\leq \sup\{\mu_B(x, y)\&_{\mu_A} \mu_B(y); y \in U) \}
\]

\[
= \mu_{IR_B}(x),
\]

\[
\nu_{IR_{B_1 \cup B_2}}(x) = \inf\{\nu_B(x, y) \&_{\nu_A} \nu_B(y); y \in U) \}
\]

\[
\geq \inf\{\nu_B(x, y) \&_{\nu_A} \nu_B(y); y \in U) \}
\]

\[
= \nu_{IR_B}(x).
\]
Not all properties of rough set operators can be generalised to multi-adjoint intuitionistic fuzzy rough set. In fact, the dualities of the lower and upper approximations are not maintained. In multi-adjoint intuitionistic fuzzy rough sets, this property is also not true. Here, we give a sufficient condition to meet this property.

Theorem 1: Let IFIS = (U, A, V, f), B ⊆ A, if μB(x, y) + νB(x, y) = 1 for all (x, y) ∈ U × U, then

\( (\mu_{Rg}(A)) = \mu_{Rg}(\sim A), \quad \sim \mu_{Rg}(\sim A) = \mu_{Rg}(\sim A) \).

Proof: According to Definition 8

\[
\begin{align*}
\mu'_{Rg} - A(x) &= \sup\{\mu(y, x) \& \mu_{\sim A}(y) : y \in U\} \\
\nu'_{Rg} - A(x) &= \inf\{1 - \nu(y, x) \& \nu_{\sim A}(y) : y \in U\}
\end{align*}
\]

From Definition 4(3), \( (\mu_{Rg}(A)) = \mu_{Rg}(\sim A) \). Similarly, \( \sim \mu_{Rg}(A) = \mu_{Rg}(\sim A) \). □

5 Attribute reduction in multi-adjoint IFIS

5.1 Multi-adjoint IF positive region

Attribute reduction in IISs has been studied extensively. Then, a large variety of algorithms for acquiring reducts are formulated [14, 35, 36]. For an IFDIS = (U, A ∪ D, V, f), where D = {d} and d ∉ A is a decision attribute, since each component of IFISs is composed of the membership and non-membership, the structure of the positive region is different from the multi-adjoint fuzzy rough set. We need to define a new form of intuitionistic fuzzy positive region dependence for IFDIS.

Definition 10: Let IFDIS = (U, A D, V, f), B ⊆ A. The multi-adjoint B-intuitionistic fuzzy rough positive region is defined as, for any y ∈ U

\[
\text{POS}_B(D)(y) = \{\mu_{\text{POS}_B(D)}(y), \nu_{\text{POS}_B(D)}(y)\}
\]

where

\[
\begin{align*}
\mu_{\text{POS}_B(D)}(y) &= \sup_{z \in U} \inf\{\mu(y, x) \& \mu(z, y)\} \\
\nu_{\text{POS}_B(D)}(y) &= \inf_{z \in U} \sup\{1 - \nu(y, x) \& \nu(z, y)\}
\end{align*}
\]

if 0 ≤ \( \mu_{\text{POS}_B(D)}(y) + \nu_{\text{POS}_B(D)}(y) \) ≤ 1; otherwise

\[
\mu_{\text{POS}_B(D)}(y) = \frac{\mu_{\text{POS}_B(D)}(y)}{\mu_{\text{POS}_B(D)}(y) + \nu_{\text{POS}_B(D)}(y)}
\]

Moreover

\[
\nu_{\text{POS}_B(D)}(y) = \frac{\nu_{\text{POS}_B(D)}(y)}{\mu_{\text{POS}_B(D)}(y) + \nu_{\text{POS}_B(D)}(y)}
\]

If R is a crisp relation, then

\[
\mu_B(x, y) = \begin{cases} 1, & \text{if } x \in [y]_B \\ 0, & \text{otherwise} \end{cases}, \quad \nu_B(x, y) = \begin{cases} 0, & \text{if } x \in [y]_B \\ 1, & \text{otherwise} \end{cases}
\]

Thus, for y ∈ U, the equations above degrade into

\[
\begin{align*}
\mu'_{\text{POS}_B(D)}(y) &= \inf_{z \in U} \{\mu_B(y, x) \& \mu_B(z, y)\} \\
\nu'_{\text{POS}_B(D)}(y) &= \sup_{z \in U} \{1 - \nu_B(y, x) \& \nu_B(z, y)\}
\end{align*}
\]

Example 3: Continuing Example 2, we expand the IFIS = (U, A, V, f), in which a decision attribute d is added as shown in Table 3. The context is (U, U, IR, t), where \( \tau \) maps all pairs of \( U \times U \) to \&. The multi-adjoint A-intuitionistic fuzzy rough positive region is

\[
\text{POS}_A(D) = \{(0.00, 0.65), (0.00, 0.64), (0.00, 0.65), (0.00, 0.75), (0.00, 0.75), (0.00, 0.89), (0.00, 0.89)\}
\]

If \( \tau \) maps all pairs of \( U \times U \) to \&L, the multi-adjoint A - the intuitionistic fuzzy rough positive region is

\[
\text{POS}_A(D) = \{(0.40, 0.60), (0.41, 0.59), (0.40, 0.60), (0.25, 0.75), (0.25, 0.75), (0.12, 0.88), (0.12, 0.88)\}
\]

5.2 Attribute reduction of L-measure

Now, we present an approach to obtain multi-adjoint intuitionistic fuzzy decision reducts. We assume that \( (L, \leq) \) is a complete lattice. From Definition 10, a definition of the degree of intuitionistic fuzzy dependence is given now.

Definition 11 (Degree of IF dependence): Let \( A \in F(U) \) the cardinality of A is defined as \( |A| = \sum_{y \in U} A(y) \). The degree of IF dependence is defined as

\[
\kappa_B = \frac{\mu_{\text{POS}_B(D)}}{\nu_{\text{POS}_B(D)}}
\]

Definition 12 (IF L-measure): \( \forall B \subseteq A \), we define a map

\[
\gamma_B : \mathcal{P}(A) \to L \text{ as } \gamma_B = \kappa_B, \quad (17)
\]

where I is a fuzzy implication.

Definition 13: Let \( \gamma : \mathcal{P}(A) \to L \) be an IF L-measure in IFDIS = (U, A D, V, f). For \( B \subseteq A, a \in L \) and \( a \neq \top_L \). If \( \gamma(B) \geq a \), B is an IF supper reduct to a degree \( a \), Let B be an IF supper reduct to a degree \( a \), B is called as an IF decision reduct to degree \( a \), if for all \( B \subseteq B, \gamma(B) < a \).

Table 3 IFDIS

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
<th>( x_6 )</th>
<th>( x_7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a_1</td>
<td>0.90,0.05</td>
<td>0.90,0.05</td>
<td>0.10,0.70</td>
<td>0.00,0.90</td>
<td>0.10,0.80</td>
<td>0.30,0.60</td>
</tr>
<tr>
<td>a_2</td>
<td>0.70,0.20</td>
<td>0.20,0.60</td>
<td>0.10,0.80</td>
<td>0.90,0.10</td>
<td>0.10,0.90</td>
<td>0.20,0.70</td>
</tr>
<tr>
<td>a_3</td>
<td>0.20,0.80</td>
<td>0.20,0.50</td>
<td>0.10,0.80</td>
<td>0.90,0.10</td>
<td>0.10,0.90</td>
<td>0.10,0.00</td>
</tr>
<tr>
<td>a_4</td>
<td>0.07,0.20</td>
<td>0.10,0.80</td>
<td>0.90,0.05</td>
<td>0.80,0.10</td>
<td>0.80,0.05</td>
<td>0.10,0.90</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
Input: $IFDIS = (U, A \cup \{d\}, V, f)$, $A = \{a_1, a_2, \ldots, a_m\}$, $\kappa_\alpha, \kappa_\beta, \kappa_\gamma, \kappa_\lambda, \kappa_\tau$.
Output: Attribute reduct (short for $Re$).
1: Set $Re = \emptyset$
2: for $a_i \in A$ do
3: compute $\text{Sig}(A, a_i) = \kappa_\alpha - \kappa_\beta - \kappa_\gamma$
4: end for
5: while $\gamma_{a_i} < \alpha$ do
6: choose any $a_i \in A - Re$
7: if $\text{Sig}(A, a_i) > \text{Sig}(A, a_i)$ then
8: set $Re := Re \cup \{a_i\}$
9: end if
10: end for
11: end while
12: Return $Re$

Fig. 1 Algorithm 1: A heuristic algorithm to acquire attribute reduct

Theorem 2: Let $IFDS = (U, A \cup \{d\}, V, f)$, for $B, C \subseteq A$, if $B \subseteq C$, then $\gamma_B \leq \gamma_C$

Proof: Straightforward.

Example 4: Consider $IFDS = (U, A \cup \{d\}, V, f)$ in Example 3. Let $\zeta = \frac{1}{x^2 + y^2}$ and $e = \min \{1, 1 + x^2 - y^2\}$.

The degree of dependence to intuitionistic fuzzy indiscernibility relations $I_{\alpha} = I_{\alpha} - I_{\alpha} - I_{\alpha} - I_{\alpha} = I_{\alpha} - I_{\alpha}$, $I_{\alpha} - I_{\alpha}$, $I_{\alpha}$, $I_{\alpha}$, $I_{\alpha}$, $I_{\alpha} - I_{\alpha}$, $I_{\alpha}$, $I_{\alpha}$, $I_{\alpha}$, $I_{\alpha}$, $I_{\alpha}$, $I_{\alpha}$, $I_{\alpha}$, $I_{\alpha}$, $I_{\alpha}$, $I_{\alpha}$, $I_{\alpha}$, $I_{\alpha}$, $I_{\alpha}$.

According to Definition 14, let $L = \kappa_\lambda$, we obtain $\gamma_{a_1} = 0.25, \gamma_{a_2} = 0.75, \gamma_{a_3} = 0.90, \gamma_{a_4} = 0.97, \gamma_{a_5} = 0.99, \gamma_{a_6} = 0.86, \gamma_{a_7} = 0.53, \gamma_{a_8} = 0.5, \gamma_{a_9} = 0.45, \gamma_{a_{10}} = 0.25, \gamma_{a_{11}} = 0.34$.

Consequently, we claim for Example 4 that $\{a_1, a_2, a_3\}$ is a decision reduce to degree $\alpha = 0.90$, $\{a_1, a_2, a_4\}$ and $\{a_1, a_2, a_3\}$ are two decision reduces to degree $\alpha = 0.80$.

5.3 Algorithm

In this section, a heuristic algorithm is introduced to compute the attribute reduce. After that, an example is employed to demonstrate the processing of the algorithm.

Definition 14: Let $IFDIS = (U, A \cup \{d\}, V, f)$, $A = \{a_1, a_2, \ldots, a_m\}$. The significance measure of $a_i$ according to $A$ is defined by

$$\text{Sig}(A, a_i) = \kappa_\alpha - \kappa_\beta - \kappa_\gamma$$

$\text{Sig}(A, a_i)$ reflects the extent to which the dependency function of $A$ respect to $d$ decreases after removing the attribute $a_i$.

According to the above significance measure, we present a heuristic algorithm to compute attribute reduction. The process is outlined in Algorithm 1 (see Fig. 1).

In Algorithm 1 (Fig. 1), steps 2–4 are to compute the significance measure of each attribute. In steps 5–13, the attributes that have higher significance are heuristically added to Re until Re becomes the reduce. To understand the mechanism of Algorithm 1 (Fig. 1) more clearly, Example 6 is employed to illustrate the processing of the algorithm above.

Example 5: Consider the attribute reduce problem in Example 4

Step 1: Set $Re = \emptyset$
Step 2–4: Computing the significance of each attribute according to $A$

$$\text{Sig}(A, a_1) = 1.245 - 0.971 = 0.274, \text{Sig}(A, a_2) = 1.245 - 0.974 = 0.271, \text{Sig}(A, a_3) = 1.245 - 1.154 = 0.091, \text{Sig}(A, a_4) = 1.245 - 1.066 = 0.1798$$

Step 5–13: Ranking the attributes by significance, adding the attribute which has a higher significance to Re until Re becomes the attribute reduce

$$\text{Sig}(A, a_i) > \text{Sig}(A, a_2) > \text{Sig}(A, a_3) > \text{Sig}(A, a_4)$$

Adding $a_1$ to Re, $\gamma_{a_1} = 0.23 < 0.9$, adding $a_2$ to $\{a_1\}$, $\gamma_{a_2} = 0.75 < 0.9$, adding $a_3$ to $\{a_1, a_2\}$, $\gamma_{a_3} = 0.91 > 0.9$.

Thus, $\{a_1, a_2, a_3\}$ is an intuitionistic fuzzy decision reduce to a degree $\alpha = 0.90$.

6 Conclusion

Using a pair of $t$-norms and its residuals’ implication to calculate the intuitionistic fuzzy upper and lower approximations, this paper constructs a multi-adjoint intuitionistic fuzzy rough set model by establishing a relationship between objects. An important feature of the model is that the user may represent preferences among the objects in an intuitionistic fuzzy decision system. After that, a kind of approximate IF decision reduce to a degree $\alpha$ in IFDIS is proposed. According to the significance of attributes, we propose a heuristic algorithm to compute the attribute reduction. At last, an example is employed to illustrate the processing of the algorithm.

In the future, we will study the structure and reduction of the multi-adjoint intuitionistic fuzzy concept lattice.

7 Acknowledgments

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8 References


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