Resolving range–Doppler coupling in LFM waveforms by steady-state filters

Negar Najian†, Mohammad Ali Sebt†
†Faculty of Electrical Engineering, K.N. Toosi University of Technology, Seyed-Khandan bridge, Shariati Avenue, Tehran, Iran
E-mail: negar.najiyan@gmail.com

Abstract: When linear frequency modulation (LFM) waveforms are utilised to estimate target range, the error due to the velocity effect on range estimation must be reduced. This error, which is caused by the Doppler effect of target velocity, can be decreased using a Kalman filter. However, as the computational work of Kalman filters is so much, steady-state filters are applied. In this study, in order to reduce range estimation error, the coefficients of steady-state filters are calculated using two methods: first, it is computed considering the Doppler error of target velocity and with the purpose of improving the steady-state response of the filters. This method, by omitting the velocity Doppler Effect, the bias value and the root-mean-square error of the range estimation are diminished considerably. In the second method, a transfer function is obtained in order to estimate the range using Z-transform, and the coefficients are calculated with the aim of improving the transient response of the filters, and it is observed that the overshoot value and the fluctuations of the transient response of the filters reduce significantly. Furthermore, in this method, unlike the previous one, there is no need for the data of acceleration and the variations of the movement acceleration along the path.

1 Introduction

LFM waveforms are widely used in radar systems. In the radial coordinate, the tracking precision is improved by the range–Doppler coupling of LFM waveforms. One of the problems of utilising LFM waveforms is the error in range estimation owing to the Doppler effect caused by velocity. One of the solutions to reduce the error due to the velocity effect in range estimation is using steady-state filters such as $\alpha - \beta$ and $\alpha - \beta - \gamma$ filters. The inputs of these filters are the data obtained via the processing of LFM signals by the sensor. Then, this data is processed by steady-state filters in order to reduce the error caused by velocity effect and measurement noise. State equation and measurement equation are defined for steady-state filters. The measurement equation is obtained from the measurement data, and the state equation is a model that the estimated parameter satisfies it.

In steady-state filters, filter gain is considered to be constant and does not require to be updated in each step, and in the result, the computational complexity is diminished considerably comparing with Kalman filter. The gain coefficients of steady-state filters are computed according to steady-state conditions and using a parameter called tracking an index which is a function of process noise variance $\sigma_v^2$, measurement noise variance $\sigma_n$ and sampling time $T$. The gain coefficients and tracking index of $\alpha - \beta$ and $\alpha - \beta - \gamma$ filters are obtained without considering the Doppler effect of velocity on range estimation in LFM waveforms [1–6], but in practise, these equations cannot be used for the range and velocity estimations of the target, as Doppler effect of the velocity has not been included in them. The tracking index of $\alpha - \beta$ filter has been obtained considering velocity Doppler effect on range estimation [1–7]. In this paper, in Sections 2–4, the relation between $\alpha$ and $\beta$ is obtained and analysed. Also, gain coefficients of $\alpha - \beta$ filter are obtained using a tracking index and considering the Doppler effect for both positive and negative Doppler conditions and maximum values of $\alpha - \beta - \gamma$ filters are obtained considering velocity Doppler effect on range estimation for both positive and negative Doppler condition. Here, $\alpha - \beta - \gamma$ filter considers the target movement acceleration, and by estimating the acceleration of the target movement in each moment and considering the acceleration variations as a process noise provides a much accurate estimation of the velocity in comparison with $\alpha - \beta$ filter. Consequently, by eliminating velocity Doppler effect on range estimation, the bias (lag), variance and the root-mean-square error (RMSE) of target range estimation decrease considerably comparing with $\alpha - \beta$ filter. As mentioned above in tracking index method $\alpha_{opt}$ is a function of $\sigma_v^2$, and in the result, choosing an appropriate value for $\sigma_v$ is very important to reach $\alpha_{opt}$. However, in none of the available references, there is a suggestion to select a suitable value for $\sigma_v$, except in [7].

If $\alpha$, $\sigma_v$ and $\Delta t_{aoa}$ denote acceleration, maximum acceleration and maximum acceleration variations of target movement in a time period ($T$), respectively, then, $\sigma_v$ values for $\alpha - \beta$ and $\alpha - \beta - \gamma$ filters are suggested to be in $[\sigma_v/2] < \sigma_v < [\sigma_v]$ and $(\Delta t_{aoa}/2) < \sigma_v < \Delta t_{aoa}$ ranges, respectively.

Consequently, in order to select an appropriate value for $\sigma_v$, we must already know the trajectory of the target movement and guess the maximum acceleration and maximum acceleration variation. Then, we can find a proper value for $\sigma_v$ in the defined intervals using trial and error method. However, this problem becomes a big challenge when we do not know the trajectory of the target movement or when the target movement has sudden changes. So, in this paper, we decided to find a way to obtain $\alpha_{opt}$ independent of $\alpha$ and as a function of the parameters of the radar and LFM waveforms that we have designed ourselves. In other words, we were looking for a relation for $\alpha_{opt}$ which is not a function of the trajectory parameters, and is only a function of the system parameters. Another problem of improving the transient response of the filters was not investigated in any previous [8–18]. While in steady-state filters, if there is no precise preliminary guess for the target range, it takes a long time for the filter to find the target. In other words, the transient response time will be prolonged and the transient response fluctuations and overshoot value will be increased. Also, if the target has a sudden change in the trajectory, the steady-state filters will have a long delay in finding the target.

In this paper, we noted that a steady-state filter can be considered as an linear time-invariant (LTI) system, a system that is independent of input–output. So, we obtained the transfer function of the steady-state filters. Finally, we improved that this transfer function is independent of $\sigma_v$ and is only a function of the
Doppler coefficient (the parameters of LFM waveforms), \(\alpha, \beta\) and \(\gamma\). Also in LTI systems, by changing the position of the transfer function poles, the characteristics of the system's transient response can be changed. For example, by minimising the size of the complex poles, the overshoot and the length and fluctuations of the transient response can be greatly reduced [19–21]. Therefore, in order to improve the transient response, we must select \(\delta_{\text{opt}}\) value so that the size of the poles of the filter transfer function is minimised.

In Section 5, by using the Z-transform method and minimising the size of the poles of the transfer function, the transient response was significantly improved.

2 Study of velocity Doppler effect on target range estimation in LFM waveforms

In this section, the error due to velocity Doppler effect on range estimation in LFM waveforms is described. Fig. 1 shows transmitted and received LFM waveforms for a moving target. In this method, using stretch processing, the transmitted and received signals have been multiplied by each other, and a single-tone signal has been obtained that its frequency is equal to the frequency difference between the transmitted and received signals.

If the target range is \(R\), and the velocity of LFM waveform is \(c\), then, \(t_d\), the time delay between the transmitted and received waveforms is equal to

\[
t_d = \frac{2R}{c}
\]

(1)

If \(m\) (Hz/s) is the slope of the LFM waveform, the frequency difference between the transmitted and received signals, without considering the Doppler effect is equal to

\[
\Delta f = \frac{2Rm}{c} \times m
\]

(2)

Target velocity due to Doppler effect causes a frequency deviation in the received waveform which is called Doppler frequency and is denoted by \(f_d\). In the following, it is illustrated how Doppler frequency causes an error in range estimation.

If the target velocity is \(V\), and the wavelength of the transmitted waveform is equal to \(\lambda\), then, the Doppler frequency is equivalent to

\[
f_d = \frac{2V}{\lambda}
\]

(3)

So, the measured frequency difference between the transmitted and the received waveforms is as

\[
\Delta f' = \Delta f + f_d = \frac{2Rm}{c} + \frac{2\nu}{\lambda}
\]

(4)

\[\Delta f' = \frac{2R'}{c}\]

(5)

\(R'\) is the estimated range and is as below by replacing (5) in (4):

\[
R' = \frac{c}{2m}f_d' = \frac{c}{2m}(\Delta f + f_d)
\]

(6)

\[
R' = \frac{c}{2m}\left(\frac{2Rm}{c} + \frac{2\nu}{\lambda}\right) = R + \left(\frac{c}{2m}\right)V
\]

(7)

In (7), \(R\) is the real range and \(V\) (c/\(\lambda m\)) is the range calculating error due to Doppler Effect, and as it was proved, this error is a coefficient of target velocity. Subsequently, in order to decrease the velocity effect on range estimation, steady-state filters, named as \(\alpha - \beta\) and \(\alpha - \beta - \gamma\) filters, are used.

3 Mathematical model

In steady-state filters such as Kalman filter, state and measurement equations are defined as follows.

State equation

\[
X_{n+1} = F_nX_n + G_nw_n
\]

(8)

where \(X_n\) is the state vector; \(F_n\) denotes a linear limit on system dynamics; \(G_n\) denotes the manoeuvre input; \(T\) is the sampling time of the measurements; and \(w_n\) is the process noise which is white Gaussian noise as \(w_n = N(0, \sigma^2_w)\).

Measurement equation

\[
y_n = H_nX_n + v_n
\]

(9)

where \(y_n\), \(\Delta t\), \(H_n\) and \(v_n\) represent target position measurement, range–Doppler coupling coefficient, output matrix and measurement noise (\(v_n = N(0, \sigma^2_v)\)), respectively. According to (10), it is observed that the range measured by the sensor (\(y_n\)) is equal to the sum of target's real range (\(x_n\)), sensor error (\(\delta_{\text{opt}}\)) and the error caused by Doppler effect (\(\Delta t\nu_n\)) which is a coefficient of target velocity.

According to (10) and considering (7), \(\Delta t\) is equal to

\[
\Delta t = \frac{c}{\lambda m} = \frac{f_d}{\text{BW}} \Rightarrow \frac{c}{\lambda} = f_d\quad \text{m} = \frac{\text{BW}}{\tau}
\]

(11)

In (11), \(f_d\), \(\tau\) and \(\text{BW}\) represent carrier frequency, pulse length and bandwidth of LFM waveform, respectively.

Time update equations

\[
\dot{X}_{n+1} = F_n\dot{X}_n + K_n\left(y_n - H_n\dot{X}_{n+1}\right)
\]

(12)

\[
P_{n+1} = F_nP_nF_n^T + G_nG_n^T + \sigma^2_v
\]

(13)

Measurement update equations

\[
\dot{X}_{n+1} = \dot{X}_{n+1} + K_n\left(y_n - H_n\dot{X}_{n+1}\right)
\]

(14)

\[
P_{n+1} = \left(I - K_nH_n\right)P_{n+1}
\]

(16)

\(\dot{X}_{n+1}\) represents the parameter estimation vector at the time \(t_n\) from measurements up to time \(t_j\); \(P_{n+1}\) is the covariance of estimation error \(\dot{X}_{n+1}\) and \(K_n\) is the filter gain in time \(t_n\). [6]

Fig. 1 Transmitted and received LFM waveform for a moving target
4 Obtaining the tracking index and gain coefficients of steady-state filters

Steady-state filters such as Kalman filter in steady state, satisfy the equations \( k_n = k_{n-1} \), \( P_n = P_{n-1} \) and \( P_{n-1} = P_{n-2} \). In the Kalman filter, in order to reach the steady-state conditions, process noise \( \sigma_w \) and measurement noise \( \sigma_v \) must have stationary statistics, and the data rate must be constant. If process noise and measurement noise do not have stationary statistics and the data rate is not constant, then optimal estimation will not be achievable using steady-state filters [7].

4.1 Obtaining the \( \alpha - \beta \) filter tracking index

Now, by utilising the equations that are satisfied in a steady state, the tracking index equation for \( \alpha - \beta \) filter is obtained

\[
P_{n-1} = P_{n-1} - P_{n-1}^T = P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}
\]

(17)

Using (13), we have

\[
X_n, F_n, G_n, H_n \quad \text{and} \quad K_n \quad \text{for} \quad \alpha - \beta \quad \text{filters are defined as follows and} \quad X_n \quad \text{is the state vector which consists of} \quad x_n \quad \text{and} \quad x_{n'} \quad \text{that represent the range and velocity of the target at time} \quad n, \quad \text{respectively, and} \quad T \quad \text{is the sampling time of the measurements}
\]

\[
X = [x_n, x_{n'}]^T
\]

(18)

\[
F_n = F = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}
\]

(19)

\[
G_n = G = \begin{bmatrix} T^2 \ & T \\ 1 \ & \Delta t \end{bmatrix}
\]

(20)

\[
H_n = H = [1 \ & \Delta t]
\]

(21)

\[
K_n = K = \begin{bmatrix} \alpha \ & \beta \end{bmatrix}^T
\]

(22)

Using (13), we have

\[
P_{n-1} = F^T[P_{n-2} - G^T\beta_n^2G](F^T)^T
\]

(23)

By replacing \( P_{n-1} \) from (16) and \( P_{n-2} \) from (23) both in (17), (24) is attained

\[
|I - KH|P_{n-1} = F^T[P_{n-2} - G^T\beta_n^2G](F^T)^T
\]

(24)

To solve (24), first, \( K_n \) is obtained. If it is assumed that the equations limit \( n \rightarrow \infty \) \( P_{n-2} = [p_{n}] \) and limit \( m_{n} \rightarrow \infty \) \( P_{n-1} = [m_{n}] \) are satisfied in steady state, then, according to (15) and considering steady-state conditions, (27) and (28) are obtained for filter gain \( K_n \) (\( P_{n-2} \) and \( P_{n-1} \) are symmetric matrices)

\[
\lim_{n \rightarrow \infty} K_n = K = [\alpha \ & \beta \ T] = [g_{1} \ & g_{1}]
\]

(25)

\[
g_{1} = \frac{m_{1} + \Delta m_{1}}{m_{1} + 2\Delta m_{1} + \Delta \Delta m_{1} + \sigma_{v}^2}
\]

(27)

\[
g_{2} = \frac{m_{2} + \Delta m_{2}}{m_{2} + 2\Delta m_{2} + \Delta \Delta m_{2} + \sigma_{v}^2}
\]

(28)

By solving (24), and using (27), the equation of tracking index (29) is achieved (see Appendix 9.1)

\[
T = \frac{T' \sigma_{n}^2}{\sigma_{v}^2}
\]

(29)

\[
\Gamma = \frac{\beta}{1 - \alpha - \beta(\Delta t/T)}
\]

(30)

\[
\left(\frac{\Delta t}{T} + 1\right)\beta^2 + \left(2\alpha + 2\frac{\beta}{(1 - \alpha(\Delta t/T))^2} - 2\alpha\right)\beta + \alpha^2 = 0
\]

(31)

There are two solutions for (31) and the valid solutions for \( \beta \) are given by

\[
\beta_n = \frac{2(-\alpha(1 + 2(\Delta t/T) + 2 \pm 2\sqrt{1 - \alpha(\Delta t/T)^2} - \alpha)}{(1 + 4(\Delta t/T))}
\]

(32)

With the aim of having positive value for the term under radical in (32) for \( (\Delta t/T) > 0 \), the maximum value of \( \alpha, \alpha_{\text{max}} \), is equal to

\[
\alpha_{\text{max}} = \frac{2(\Delta t/T) + 1 + \sqrt{4(\Delta t/T) + 1}}{2(\Delta t/T)}
\]

(33)

Two answers are evaluated for beta in (32). In the condition \( (\Delta t/T) > 0 \), both \( \beta \) and \( \beta_n \) are valid; \( \beta_n \) for the case that \( \alpha \) increases from zero to \( \alpha_{\text{max}} \) and \( \beta_n \) for the case that \( \alpha \) decreases from \( \alpha_{\text{max}} \) to \( \alpha \). To achieve \( \alpha_{\text{max}} \), \( \alpha \) is reduced to the value that the condition \( 1 - \alpha - \beta(\Delta t/T) > 0 \) is satisfied according to (30). In the condition \( (\Delta t/T) < 0 \), both \( \beta_n \) is valid, and also, in this case, \( \alpha \) is increased from zero up to the value that the condition \( 1 - \alpha - \beta(\Delta t/T) < 0 \) is satisfied according to (30). Consequently, in this case, \( \alpha \) is augmented up to the value that the inequality \( \alpha < 1 - \beta(\Delta t/T) \) is fulfilled [7].

4.2 Obtaining the \( \alpha - \beta \) filter gain coefficients

Now, by utilising (34) that is obtained from (39), \( \alpha - \beta \) filter gain coefficients are obtained considering Doppler effect for both positive and negative Doppler coefficient conditions with the purpose of estimating the range and velocity of the target

\[
\beta_n^2 + \Gamma \beta_n^\Delta t \frac{\Delta t}{T} + \Gamma^2 \alpha - \Gamma^2 = 0
\]

(34)

To omit \( \beta \) from (34), (32) is used in which \( \beta \) is described versus \( \alpha \) in the condition \( (\Delta t/T) \neq 0 \). In this step, two cases are considered for the Doppler coefficient; \( (\Delta t/T) > 0 \) and \( (\Delta t/T) < 0 \).

4.2.1 Positive Doppler coefficient \((\Delta t/T) > 0\): In this case, two answers of beta \( (\beta, \beta_n) \) in (41) are acceptable, and by substituting (32) into (41), (43) is transformed to the following two equations:

\[
\beta_n^2 + \Gamma \beta_n^\Delta t \frac{\Delta t}{T} + \Gamma^2 \alpha - \Gamma^2 = 0
\]

(35)

\[
\beta_n^2 + \Gamma \beta_n^\Delta t \frac{\Delta t}{T} + \Gamma^2 \alpha - \Gamma^2 = 0
\]

(36)

According to (35) and (36), two answers are obtained for \( \alpha, \alpha_{\text{max}} \) with the condition \( 0 < \alpha < \alpha_{\text{max}} \) from (35) and \( \alpha, \alpha_{\text{max}} \) with the condition \( \alpha_{\text{max}} < \alpha \) from (36). Thus, it can be stated that two equations are obtained for filter gain for \( (\Delta t/T) > 0 \)

\[
\beta_n \quad \Gamma \beta_n^\Delta t \frac{\Delta t}{T}
\]

(37)

\[
\beta_n \quad \Gamma \beta_n^\Delta t \frac{\Delta t}{T}
\]

(38)

Deciding between (37) and (38) to get an appropriate filter gain depends on the value of the tracking index in (30). Fig. 2 shows the
α coefficient for α − β filter versus tracking index (Γ) assuming = 0.022, in the conditions Δt/T = 1, 0.5, 0.25. In this figure, solid lines are related to α, i.e. Kα gain and dotted lines are related to αs, i.e. Ks gain. According to Fig. 2, in the condition Δt/T > 0, the value of tracking index (Γ) determines that which of Kα or Ks is valid for the filter gain.

4.2.2 Negative Doppler coefficient (Δt/T < 0): In this case, only β satisfies (34). Hence, for Δt/T < 0, only (35) is satisfied, and for the filter gain, only Ks is acceptable.

\[ K_s = \left[ \alpha \quad \beta \quad \frac{T}{T} \right] \]  \hspace{1cm} (39)

4.3 Obtaining α − β − γ filter tracking index

Now, by utilising the equations that are satisfied in steady state, the tracking index equation of α − β − γ filter is obtained (Pαβ and Pαβ−1 are symmetric matrices)

\[ P_{\alpha\beta} = P_{\alpha\beta-1} = P = \begin{bmatrix} p_1 & p_2 & p_3 \\ p_1 & p_2 & p_3 \\ 0 & 1 & T \end{bmatrix} \]  \hspace{1cm} (40)

Xn, Fn, Gn, Hn and Ks for α − β − γ filter are defined as follows, Xn is the state vector which consists of x0, x0, and x0, that represents the range, velocity and acceleration of the target at time n, respectively, and T is the sampling time of the measurements

\[ X_n = \begin{bmatrix} x_0 \\ x_0 \\ x_0 \end{bmatrix} \]  \hspace{1cm} (41)

\[ F_n = F = \begin{bmatrix} 1 & T & \frac{T^2}{2} \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \]  \hspace{1cm} (42)

\[ G_n = G = \begin{bmatrix} \frac{T^2}{2} & T & 1 \end{bmatrix} \]  \hspace{1cm} (43)

\[ H_n = H = [1 \quad \Delta t \quad 0] \]  \hspace{1cm} (44)

To solve (24) for α − β − γ filter, first, Ks is obtained. According to (46) and considering steady-state conditions, (48)–(50) are obtained for filter gain Ks

\[ \lim_{n \to \infty} K_n = K_s = \left[ \alpha \quad \beta \quad \frac{T}{T} \right] = \begin{bmatrix} \dot{g}_1 & \dot{g}_2 & \dot{g}_3 \end{bmatrix} \]  \hspace{1cm} (45)

\[ K = \begin{bmatrix} \dot{g}_1 & \dot{g}_2 & \dot{g}_3 \end{bmatrix}^T = P_{\alpha\beta\gamma-1}^T \begin{bmatrix} H \end{bmatrix} \]  \hspace{1cm} (46)

\[ K = \frac{1}{m_1 + 2\Delta tm_{12} + \Delta^2 tm_{22} + \sigma_i^2} \times \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \Delta t \]  \hspace{1cm} (47)

\[ g_1, \ g_2 \text{ and } g_3 \text{ are calculated as below according to (47):} \]

\[ g_1 = \frac{m_{11} + \Delta tm_{12}}{m_{11} + \Delta tm_{12} + 2\Delta tm_{22} + \sigma_i^2} \]  \hspace{1cm} (48)

\[ g_2 = \frac{m_{12} + \Delta tm_{22}}{m_{11} + \Delta tm_{12} + 2\Delta tm_{22} + \sigma_i^2} \]  \hspace{1cm} (49)

\[ g_3 = \frac{m_{13} + \Delta tm_{32}}{m_{11} + \Delta tm_{12} + 2\Delta tm_{22} + \sigma_i^2} \]  \hspace{1cm} (50)

In this step, the system of (51) is achieved. The derivation of (51) is given in Appendix 9.2

\[ \begin{bmatrix} \sigma_{\dot{v}}^2 \frac{\alpha^2}{T^2} \sigma_{\dot{\gamma}}^2 \frac{\alpha^2}{T^2} \sigma_{\dot{\beta}}^2 \frac{\alpha^2}{T^2} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \sigma_{\dot{v}}^2 \frac{\alpha^2}{T^2} \sigma_{\dot{\gamma}}^2 \frac{\alpha^2}{T^2} \sigma_{\dot{\beta}}^2 \frac{\alpha^2}{T^2} \end{bmatrix} \]  \hspace{1cm} (51)

If the first and second equations of (51) are multiplied by β and αT, respectively, by subtracting the two equations, (52) will be obtained

\[ \gamma = \frac{\dot{\beta}}{\alpha} \]  \hspace{1cm} (52)

Moreover, if the first equation of (51) is multiplied by \( T^4/\sigma_{\dot{v}}^2 \), (53) will be obtained

\[ \frac{T^4\sigma_{\dot{v}}^2}{\sigma_{\dot{\gamma}}^2} \alpha^2 \beta + \frac{T^4\sigma_{\dot{v}}^2}{\sigma_{\dot{\beta}}^2} \alpha^2 \gamma + \frac{1}{4} \alpha \beta \gamma^2 = 0 \]  \hspace{1cm} (53)

Now, (53) is arranged versus tracking index, and leads to (54)

\[ \Gamma^2 \alpha^2 \beta + \Gamma \left( \frac{\Delta t}{T} \right) \Gamma \gamma - \Gamma^2 \alpha \beta = \frac{1}{4} \alpha \beta \gamma^2 = 0 \]  \hspace{1cm} (54)

By utilising (54), the tracking index (Γ) for α − β − γ filter is attained when \( \Delta t/T \neq 0 \)

\[ \Gamma^2 = \frac{T^4\sigma_{\dot{v}}^2}{\sigma_{\dot{\gamma}}^2} = \frac{\gamma^2}{4(1 - \alpha - \beta(\Delta t/T))} \]  \hspace{1cm} (55)

4.4 Obtaining the α − β − γ filter gain coefficients

In this step, according to (55) and (32), the gain coefficients of α − β − γ filter are obtained, considering the Doppler effect for both positive and negative Doppler coefficient conditions. By substituting (52) into (55) we have

\[ \Gamma^2 = \frac{\beta^2}{4(1 - \alpha - \beta(\Delta t/T))} \]  \hspace{1cm} (56)

\[ 4\Gamma^2 \alpha^2 - 4\Gamma^2 \alpha^2 \beta \Delta t + \beta^4 = 0 \]  \hspace{1cm} (57)

To omit β from (57), (32) is used in which β is described versus α in the condition of Δt/T ≠ 0.

4.4.1 Positive Doppler coefficient (Δt/T > 0): In this case, by substituting (32) into (57), (57) is transformed to the following two equations:

...
\[4\alpha T^2 - 4\alpha T^2 + 4 \Delta t \frac{\alpha}{T} \beta_1 T^2 + \beta_2^2 = 0 \quad (58)\]

\[4\alpha T^2 - 4\alpha T^2 + 4 \Delta t \frac{\alpha}{T} \beta_1 T^2 + \beta_2^2 = 0 \quad (59)\]

According to (58) and (59), two answers are obtained for \(\alpha, \alpha_0\) with the condition \(a_{\text{min}} < \alpha, \alpha_0 < a_{\text{max}}\) from (58), and \(\alpha, \alpha_0\) with the condition \(0 < \alpha, \alpha_0 < a_{\text{max}}\) from (59). Therefore, there exist two answers for \(\gamma\) as well as according to (52)

\[y_+ = \frac{\beta_1}{\alpha} \quad (60)\]

\[y_- = \frac{\beta_1}{\alpha} \quad (61)\]

Thus, two equations are obtained for filter gain in the condition \(\Delta t/T > 0\)

\[K_+ = \begin{bmatrix} \alpha & \beta_1 T & y_+ 27^T \end{bmatrix} \quad (62)\]

\[K_- = \begin{bmatrix} \alpha & \beta_1 T & y_- 27^T \end{bmatrix} \quad (63)\]

Deciding between (62) and (63) to get an appropriate filter gain depends on the value of tracking index in (55). Fig. 3 shows the \(\alpha\) coefficient for \(\alpha = \beta - \gamma\) filter versus tracking index \((\Gamma)\) assuming \(-0.0222, \text{in the conditions } \Delta t/T = 1, 0.5, 0.25\). In this figure, solid lines are related to \(\alpha, \text{i.e. } \Gamma\) gain, and dotted lines are related to \(\alpha_0, \text{i.e. } \Gamma\) gain. According to Fig. 3, for \(\Delta t/T > 0\), the value of tracking index \((\Gamma)\) determines that which of \(K_+\) or \(K_-\) is valid for the filter gain.

### 4.4.2 Negative Doppler coefficient \((\Delta t/T < 0):\)

In this section, \(\alpha\) is obtained utilising Z-transform and by defining transfer function for steady-state filters in range estimation such that the amplitude of the transfer function poles, and consequently, the overshoot \((\text{OV})\) value and the fluctuations of the transient response of the filters in range estimation are minimised.

### 5 Calculation of \(\alpha\) using Z-transform

In this section, \(\alpha\) is obtained in order to improve the transient response of range estimation in \(\alpha = \beta - \gamma\) filter. If \(X_{n}\) is the vector of parameter estimation in \(\alpha = \beta - \gamma\) filter, then, we have

\[X_{n} = [x_{0} \quad x_{n}]^T \quad (65)\]

\[X(0) = [x(0) \quad x(0)]^T = [0 \quad 0]^T \quad (66)\]

For target movement with constant acceleration, we have

\[x_{n} = a(nT)^2 + a(nT) x(0) + x(0) = a(nT)^2 \quad (67)\]

\[x_{n} = a(nT) x + x(0) = a(nT) \quad (68)\]

By substituting (67) and (68) into (9) and without considering the measurement noise \((\sigma^2)\), it is achieved [19]

![Fig. 3](image)

\[\alpha \text{ Coefficient for } \alpha = \beta - \gamma \text{ filter versus tracking index}\]

\[Y_n = [1 \Delta T] \begin{bmatrix} \alpha \frac{a(nT)^2}{2} \\ x_n = a(nT) \end{bmatrix} \quad (69)\]

\[Y_n = a(nT)^2 + a(nT) \Delta t \quad (70)\]

By replacing (12) into (14), (71) is obtained as follows:

\[X_{an} = [I - KH] X_{an-1} + K X_{n} - K X_{n} \quad (71)\]

\[\vec{G} = K = \begin{bmatrix} \alpha \beta T \end{bmatrix} \quad (73)\]

\[\vec{F} = [I - \bar{K}H] F = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & \alpha \beta \bar{T} & \bar{T} \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} \quad (74)\]

\[X_{an} = \bar{F} X_{n-1} + \bar{G} Y_{n} \quad (76)\]

If (71) is transformed by Z-transform, the following equations will be obtained:

\[\hat{X}(z) = \bar{F} \hat{X}(z) \quad (77)\]

\[\hat{X}(z) = [I - \bar{F} \bar{Z}^{-1}] \bar{G} Y_{n} \quad (78)\]

\[\hat{X}(z) = [X(z) \quad \hat{X}(z)] \quad (79)\]

If (70) is transformed by Z-transform, (80) will be achieved as below:

\[Y(z) = a(nT)^2 (Z + 1) + a(nT) a(nT) \quad (80)\]

Now, if \(Y(z)\), which is the Z-transform of the observations or the range measured by the sensor, is considered as the filter input, and \(X(z)\), which is the Z-transform of the estimated range by filter, is considered as the filter output, then, by dividing the filter output \([X(z)]\) into its input \([Y(z)]\), an equation named as the transfer function (transfer function) of the range estimation in \(\alpha = \beta - \gamma\) filter is obtained. Thus, by dividing (78) into (79), we have
By substituting (80) into (81), it is achieved as follows:

\[
\frac{1}{Y(z)} = [I - \tilde{F}Z^{-1}]^{-1} \tilde{G} \tag{82}
\]

By calculating the matrix obtained in (83), the first row of the matrix is considered as the transfer function of range estimation in the filter, and is denoted by \( M(z) \)

\[
M(z) = \frac{X(z)}{Y(z)} = \frac{Z(a(Z-1) + \beta)}{T(Z^2 + (a + \beta(1 + (\Delta t/T)) - Z^2 + 1 - a - \beta(1 + (\Delta t/T)))} \tag{84}
\]

The transient response of a digital control system depends on the location of the poles of its transfer function on the Z-plane. When the transfer function of a stable system has complex poles inside the unit circle on the Z-plane, the transient response of the system is oscillatory and has positive damping. In other words, the fluctuations of the transient response will be disappeared over time. In general, for the transfer function of a stable system, whichever the complex poles, which are within the unit circle, get closer to the unit circle, the transient response of the system will be more oscillating, and the OV value will increase. Hence, by decreasing the amplitude of the complex poles of the transfer function, we move them far from the unit circle and make them closer to the zero. In result, the system becomes more stable, and the length and the fluctuations of its transient response, and also the amount of the OV decreases, and the convergence speed to steady-state response increases.

In this step, the poles of the transfer function of range estimation in (84), i.e. \( Z_{c1} \) and \( Z_{c2} \), are computed

\[
Z_{c1, c2} = \frac{1}{2} \left( 2 - a - \beta \left( 1 + \frac{\Delta t}{T} \right) \right) \\
\pm \frac{i}{2} \sqrt{\left( 1 - a - \beta \left( 1 + \frac{\Delta t}{T} \right) \right) - \left( 2 - a - \beta \left( 1 + \frac{\Delta t}{T} \right) \right)} \tag{85}
\]

Now, the value of \( \alpha \) is chosen such that the amplitude of the poles of the range estimation transfer function in (85) is minimised

\[
\alpha_{c1} = \text{Arg} \arg_{min} \left| Z_{c1} \right| \tag{86}
\]

Now, by replacing (32) in (86), for \( \Delta t/T > 0 \), between \( \beta_{c1} \) and \( \beta_{c2} \), the one is selected to replace which minimises the amplitude of the poles of the transfer function. However, for \( \Delta t/T < 0 \), only \( \beta_{c2} \) can be replaced in (84) from (32).

5.2 Calculation of \( \alpha \) in \( \alpha - \beta - \gamma \) filter using Z-transform

In this section, in order to calculate \( \alpha \) in \( \alpha - \beta - \gamma \) filter using Z-transform, the transfer function of range estimation in \( \alpha - \beta - \gamma \) filter is obtained

\[
X_n = [x_n \ \dot{x}_n \ \ddot{x}_n]^T \tag{87}
\]

Equation (88) is rewritten for \( \alpha - \beta - \gamma \) filter as below:

\[
Y_n = \begin{bmatrix} 1 & \Delta t & 0 \end{bmatrix} \begin{bmatrix} x_n \ a(\Delta t) \ \dot{x}_n \end{bmatrix}^T \tag{89}
\]

With the aim of attaining the transfer function of range estimation, (81) must be obtained for \( \alpha - \beta - \gamma \) filter

\[
\frac{1}{Y(z)} \times \tilde{X}(z) = [I - \tilde{F}Z^{-1}]^{-1} \tilde{G} \tag{90}
\]

In this step, the matrix of (90) is calculated for \( \alpha - \beta - \gamma \) filter

\[
\tilde{F} = [I - \tilde{F}Z^{-1}] = \begin{bmatrix} 1 & \alpha \ \beta \ \gamma \end{bmatrix} \tag{91}
\]

To compute the inverse of the matrix of (94), first, the adjoint of the matrix \( [I - \tilde{F}Z^{-1}] \), i.e. \( [I - \tilde{F}Z^{-1}]^{-1} \), is attained. Then, it is divided into the matrix determinant, i.e. \( [I - \tilde{F}Z^{-1}] \), as follows:

\[
[I - \tilde{F}Z^{-1}]^{-1} = \frac{1}{[I - \tilde{F}Z^{-1}]} \times [I - \tilde{F}Z^{-1}] \tag{95}
\]

Now, (90) can be written, for \( \alpha - \beta - \gamma \) filter, as below: (see (96)). As mentioned in the previous section, the first row of the matrix obtained from (90) is the transfer function of range estimation in \( \alpha - \beta - \gamma \) filter. Here, according to (96), the denominator of the transfer function of range estimation is the determinant of the matrix \( I - \tilde{F}Z^{-1} \). Consequently, it can be stated that the
denominator of the transfer function of range estimation in $a - \beta - \gamma$ filter is equal to the equation below:

$$
[I - FZ^{-1}]^{-1} = \frac{1}{I - FZ^{-1}} \begin{bmatrix}
1 - aZ^{-1} & -aTZ^{-1}(1 + \frac{\Delta t}{\gamma}) & -aT^2Z^{-1}(\frac{1}{2} + \frac{\Delta t}{T}) \\
-\frac{\beta}{T}Z^{-1} - 1 - \beta Z^{-1}(1 + \frac{\Delta t}{\gamma}) & -\frac{\beta}{T}Z^{-1}(\frac{1}{2} + \frac{\Delta t}{T}) & -\frac{\beta}{T}Z^{-1}(1 + \frac{\Delta t}{\gamma}) \\
-rac{\alpha}{\gamma}Z^{-1} - 1 - \frac{\alpha}{\gamma}Z^{-1}(1 + \frac{\Delta t}{\gamma}) & -\frac{\alpha}{\gamma}Z^{-1}(\frac{1}{2} + \frac{\Delta t}{T}) & -\frac{\alpha}{\gamma}Z^{-1}(1 + \frac{\Delta t}{\gamma})
\end{bmatrix}
$$

(96)

t_1 = A + B

(110)

$$
t_{2,3} = -\frac{1}{2}(A + B) \pm \sqrt{\frac{3}{4}}(A - B)
$$

(111)

Accordingly, the roots of (99) are equal to

$$
Z_i = A + B - \frac{b}{3a}
$$

(112)

$$
Z_{2,3} = -\frac{1}{2}(A + B) - \frac{b}{3a} \pm \sqrt{\frac{3}{4}}(A - B)
$$

(113)

In this step, $\alpha$ is obtained such that the amplitudes of the roots of $Z_i$ in (113) become minimised. Therefore, $\alpha$ is equivalent to

$$
a_{\text{opt}} = \text{Arg min}_{a_i} |Z_i|
$$

(115)

By substituting (32) and (60) or (61) into (115), as it was mentioned in the previous section, for $\Delta t/T > 0$, between $(\beta_1, \gamma_1)$ and $(\beta_2, \gamma_2)$, the one is selected to replace which minimises the amplitude of the poles of the transfer function. However, for $\Delta t/T < 0$, only $(\beta_1, \gamma_1)$ can be substituted in (115) from (32) and (61).

6 Simulation results

Target movement is considered to be two-dimensional (2D) and parabolic in each dimension. The target movements in $x$ and $y$ axes are denoted by $x[n]$ and $y[n]$, respectively. If the acceleration, the initial velocity and the initial range in the $x$-axis are as $a_x = 200(\text{m/s}^2)$, $v_{x_0} = 200(\text{m/s})$ and $x_0 = 1000\text{ m}$, respectively, the acceleration, the initial velocity and the initial range in the $y$-axis are as $a_y = 200(\text{m/s}^2)$, $v_{y_0} = 200(\text{m/s})$ and $y_0 = 2000\text{ m}$, respectively, and $R[n]$ and $V[n]$ are the range target and the range velocity in 2D space, respectively, the parabolic trajectory will be defined as below:

$$
x[n] = 0.5a_x(nT)^2 + v_{x_0}(nT) + x_0
$$

(116)

$$
y[n] = 0.5a_y(nT)^2 + v_{y_0}(nT) + y_0
$$

(117)

$$
v_x[n] = a_x(nT) + v_{x_0}
$$

(118)

$$
v_y[n] = a_y(nT) + v_{y_0}
$$

(119)

$$
V = \sqrt{v_x[n]^2 + v_y[n]^2}
$$

(120)

$$
R = \sqrt{x[n]^2 + y[n]^2}
$$

(121)

So observed range data considering Doppler effect and measurement noise ($v_n \cong N(0, \sigma_n^2)$), $Z_R$ is equal to

$$
Z_R = R + V\Delta t + v_n
$$

(122)

As in steady-state filters, the objective is reducing the RMSE value of parameter estimation, the RMSE value of range estimation for steady-state responses of two $a - \beta$ and $a - \beta - \gamma$ filters are
compared. If the range estimation is \( \hat{R} \), Bias\( (\hat{R}) \) and Var\( (\hat{R}) \) denote the bias and variance of \( \hat{R} \), respectively. Then the RMSE value of range estimation is equal to

\[
\begin{align*}
\hat{R} &= \hat{R}_{\text{meas}} - \hat{R}_{\text{true}} + \alpha [Z_{\text{true}} - \hat{R}_{\text{true}} - V_a \Delta t] \\
\text{RMSE}_{R} &= \sqrt{\text{Var}(\hat{R})} = \sqrt{(\text{Bias}(\hat{R}))^2 + \text{Var}(\hat{R})} 
\end{align*}
\]  

(123)

(124)

In the defined parabolic trajectory, it is assumed that \( \sigma_{\text{std}} = 287.04 \) and \( \Delta \sigma_{\text{std}} = 9.89 \). Thus, 143.2 < \( \sigma_{\text{std}} = 287.04 \) and 4.94 < \( \sigma_{\text{std}} = 9.89 \) are selected for \( \alpha - \beta \) and \( \alpha - \beta - \gamma \) filters, respectively. In the following, the simulation results of the target range estimation in the time interval \( 0 \leq t \leq 15 \) s for \( f_s = 10 \) MH, \( B_{\text{W}} = 9 \) GH, \( \tau = 20 \) ms, \( T = 0.0222 \) ms, \( \Delta T / T = 1 \), \( \sigma_{\text{std}} = 10 \) m, and \( \sigma_{\text{std}} = 5 \) (m/s) for \( \alpha - \beta \) and \( \alpha - \beta - \gamma \) filter is demonstrated. The initial values of \( \alpha - \beta \) and \( \alpha - \beta - \gamma \) filters are as below, respectively

\[
X_{\text{a-\beta}} = [R(0) \ V(0)]^T = [1000 \ 100]^T \\
X_{\text{a-\beta-\gamma}} = [R(0) \ V(0) \ a(0)]^T = [1000 \ 100 \ 100]^T
\]  

(125)

(126)

Figs. 4a and b show steady-state response of target range estimation versus time for moving in parabolic trajectory for \( \alpha - \beta \) and \( \alpha - \beta - \gamma \) filters, respectively, using tracking index method with filter gain \( K = K_{\gamma} \).

Figs. 5a and b show the RMSE of steady-state response error of target range estimation versus time for moving in parabolic trajectory for \( \alpha - \beta \) and \( \alpha - \beta - \gamma \) filters, using tracking index method, for \( 5 \leq t \leq 15 \) s that filter response achieves steady state and for the filter gain \( K = K_{\gamma} \).

In next section, the OV value and the length of the transient response of steady-state filters are compared for each other for two tracking index and Z-transform methods, and as the transient response of the filters is considered, the simulations are accomplished for \( 0 \leq t \leq 5 \) s. The movement trajectory and the parameter values are as previous. Fig. 6 shows the transient response of \( \alpha - \beta \) filter for range estimation comparing two tracking indexes and Z-transform methods. In Figs. 6a and b, the transient response of the filter using tracking index and Z-transform methods are demonstrated, respectively. Figs. 6c and d illustrate the RMSE of range estimation for the transient response of \( \alpha - \beta \) filter via tracking an index and Z-transform methods, respectively.

Fig. 7 shows the transient response of \( \alpha - \beta - \gamma \) filter for range estimation comparing two tracking indexes and Z-transform methods. In Figs. 7a and b, the transient response of the filter using tracking index and Z-transform methods are demonstrated, respectively. Figs. 7c and d illustrate the RMSE of range estimation for the transient response of \( \alpha - \beta - \gamma \) filter via tracking an index and Z-transform methods, respectively.

Table 1, the coefficients of \( \alpha - \beta \) and \( \alpha - \beta - \gamma \) filters are obtained by tracking index method, in for \( \Delta T / T = 1 \) and \( \sigma_{\text{std}} = 10 \) m, and by choosing an appropriate value for \( \sigma_{\text{std}} \).

According to the value of \( T \) for both \( \alpha - \beta \) and \( \alpha - \beta - \gamma \) filters, merely the gain of \( K_{\gamma} \) is valid. Moreover, in this table, the mean of RMSE and the mean of bias and variance of range estimation are calculated for \( 5 \leq t \leq 15 \) s, i.e. from the time that the filter response achieves steady state. As it is observed in Table 1, the mean of RMSE, error variance and error bias, for range estimation is reduced considerably in \( \alpha - \beta - \gamma \) filter comparing with \( \alpha - \beta \) filter. In Tables 2 and 3, the transient response characteristics are compared for two \( \alpha - \beta \) and \( \alpha - \beta - \gamma \) filters, respectively.
Fig. 6 Comparison of the transient response of $\alpha - \beta$ filter for tracking index and Z-transform methods
(a) Transient response of $\alpha - \beta$ filter using tracking index method, (b) Transient response of $\alpha - \beta$ filter using Z-transform method, (c) RMSE of the transient response of $\alpha - \beta$ filter using tracking index method, (d) RMSE of the transient response of $\alpha - \beta$ filter using Z-transform method

Fig. 7 Comparison of the transient response of $\alpha - \beta - \gamma$ filter for tracking an index and Z-transform methods
(a) Transient response of $\alpha - \beta - \gamma$ filter using tracking index method, (b) Transient response of $\alpha - \beta - \gamma$ filter using the Z-transform method, (c) RMSE of the transient response of $\alpha - \beta - \gamma$ filter using tracking index method, (d) RMSE of the transient response of $\alpha - \beta - \gamma$ filter using the Z-transform method
Moreover, in two tables, the mean of RMSE and the mean of bias and variance of range estimation are calculated for $5 \leq t \leq 15$ s, i.e. from the time that the filter response achieves steady state. In the first row of these tables, the tracking index method, and in the second row the Z-transform method are investigated. As it is observed, for both filters, by minimising the amplitude of complex poles, i.e. $\mathcal{Z}$, their imaginary part is diminished as well, and the OV value decreases intensely. Also, using Z-transform method for steady-state filter, in addition to improving the transient response of the filters, decreases the bias of range estimation, but on the contrary increases the variance of range estimation.

7 Conclusion

This paper consists of two parts. In the first section, the relations of the steady-state filters are obtained to improve the steady-state response. In this section, in order to reduce the range estimation error in LFM waveforms, steady-state filters were utilised, and the filter coefficients were obtained considering the error of velocity Doppler via improving the steady-state and transient responses of the filters in range estimation. First, the coefficients of the filters were calculated with the purpose of improving the steady-state response of the filters in range estimation using tracking index, and by considering the equations obtained for tracking index and $\alpha-\beta-\gamma$ filter coefficients, it was proved that $\alpha-\beta-\gamma$ filter provides more precise estimation of the range by decreasing velocity Doppler Effect, and the RMSE, bias and variance of range estimation of the steady-state response was diminished significantly with respect to $\alpha-\beta$ filter. In the second section, in order to improve the transient response of the filters in range estimation, a transfer function was defined for steady-state filters utilising Z-transform, and the filter coefficients were achieved such that the amplitudes of the poles of the transfer function of range estimation were minimised, and it was observed that the OV value and the length of the transient response was reduced considerably.

This method has several advantages. First, it is not necessary to guess an exact proper initial value for the target range. With any initial guess for the range, the filter can find the target quickly. For example, in the simulations provided in this paper, the initial value of the range is assumed to be zero. Second, $\alpha_{opt}$ is not a function of $\sigma_{\nu}$, therefore, we do not need to know the movement trajectory beforehand in order to choose $\sigma_{\nu}$. Third, the target can be found rapidly even when it redirects suddenly. Fourth, $\alpha_{opt}$ value is independent of $\sigma_{\nu}$, and there is no need to select a different $\alpha$ for each target trajectory change. However, in the tracking index method, when the target trajectory changes, $\sigma_{\nu}$ and as a result a new $\alpha$ must be selected according to the acceleration of the movement change (which, of course, we do not know of). We should also have a precise preliminary guess of the target range at the beginning of the trajectory. However, the $Z$-transform effect on steady-state response is reducing the range estimation bias, and in contrast increasing the estimation variance. Since the bias value is very high in the tracking index method for the alpha–beta filter. Using the Z-transform method, the bias value decreases considerably. Although, the variance value of the range estimation increases, in general, a significant decrease in the bias value results in a reduction in the RMSE value of the steady-state error of the range estimation in the alpha–beta filter. Conversely, in alpha–beta–gamma filter, because of the amount of estimation bias in the tracking index method is very low, the reduction of the estimation bias value, using the Z-transform method, does not affect the reduction of the RMSE of the steady-state error of the range estimation in the alpha–beta–gamma filter.

8 References


Table 1 Comparison of $\alpha-\beta$ and $\alpha-\beta-\gamma$ filter's steady-state response for tracking method

<table>
<thead>
<tr>
<th>Method</th>
<th>Filter</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\sigma_{\nu}$, m/s²</th>
<th>$\Gamma$</th>
<th>Mean of RMSE, m</th>
<th>Mean of Bias($\hat{R}$), m</th>
<th>Mean of Var($\hat{R}$), m²</th>
</tr>
</thead>
<tbody>
<tr>
<td>tracking index $\alpha-\beta$</td>
<td>$\alpha - \beta$</td>
<td>280</td>
<td>0.013</td>
<td>0.141 0.0127</td>
<td>0.3820</td>
<td>8.29</td>
<td>-7.59</td>
<td>11.13</td>
<td></td>
</tr>
<tr>
<td>tracking index $\alpha-\beta-\gamma$</td>
<td>$\alpha - \beta - \gamma$</td>
<td>5</td>
<td>2.46 $\times 10^{-1}$ 0.109 0.007 4.67 $\times 10^{-1}$ 0.3820</td>
<td>2.36</td>
<td>-0.36</td>
<td>5.45</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2 Comparison of the transient response of $\alpha-\beta$ filter for two tracking index and Z-transform methods

<table>
<thead>
<tr>
<th>Method</th>
<th>Filter</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\sigma_{\nu}$, m/s²</th>
<th>$\Gamma$</th>
<th>Mean of RMSE, m</th>
<th>Mean of Bias($\hat{R}$), m</th>
<th>Mean of Var($\hat{R}$), m²</th>
</tr>
</thead>
<tbody>
<tr>
<td>tracking index $\alpha-\beta$</td>
<td>$\alpha - \beta$</td>
<td>0.141 0.0127 0.917 ± 0.076 0.920</td>
<td>198</td>
<td>1.97</td>
<td>8.29</td>
<td>-7.59</td>
<td>11.13</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z-transform $\alpha-\beta$</td>
<td>$\alpha - \beta$</td>
<td>0.349 0.598</td>
<td>0.258</td>
<td>0.258</td>
<td>0</td>
<td>0.13</td>
<td>6.02</td>
<td>-0.83</td>
<td>37.55</td>
</tr>
</tbody>
</table>

Table 3 Comparison of the transient response of $\alpha-\beta-\gamma$ filter for two tracking index and Z-transform methods

<table>
<thead>
<tr>
<th>Method</th>
<th>Filter</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\sigma_{\nu}$, m/s²</th>
<th>$\Gamma$</th>
<th>Mean of RMSE, m</th>
<th>Mean of Bias($\hat{R}$), m</th>
<th>Mean of Var($\hat{R}$), m²</th>
</tr>
</thead>
<tbody>
<tr>
<td>tracking index $\alpha-\beta-\gamma$</td>
<td>$\alpha - \beta - \gamma$</td>
<td>0.109 0.007 4.67 $\times 10^{-1}$</td>
<td>-0.0187</td>
<td>1</td>
<td>319</td>
<td>4.928</td>
<td>2.36</td>
<td>-0.36</td>
<td>5.45</td>
</tr>
<tr>
<td>Z-transform $\alpha-\beta-\gamma$</td>
<td>$\alpha - \beta - \gamma$</td>
<td>0.334 0.662</td>
<td>1.314</td>
<td>-0.0187</td>
<td>0</td>
<td>0.17</td>
<td>5.45</td>
<td>-0.104</td>
<td>27.02</td>
</tr>
</tbody>
</table>
Appendix

9.1 Derivation of (29) and (31)

The first and second sides of (24) for $\alpha - \beta$ filter are equal to (127) and (128), respectively.

\[
(I - KH)P_{\text{pre-1}} = 
\begin{bmatrix}
    m_1(1 - g_1) - m_2g_2T - m_3(1 - g_1) - g_1\Delta m_2 \\
    -g_2m_1 + m_2(1 - g_1)T - g_2m_3 + m_3(1 - g_1)T
\end{bmatrix}
\]

(127)

\[
F^{-1}[P_{\text{pre-1}} - GG^T\hat{\sigma}_w^2](F^{-1})^T =
\begin{bmatrix}
    m_1 - 2Tm_2 + T^2m_3 - \frac{T^3}{4}\hat{\sigma}_w^2 & m_2 - Tm_2 + \frac{T^3}{2}\hat{\sigma}_w^2 \\
    m_2 - Tm_2 + \frac{T^3}{2}\hat{\sigma}_w^2 & m_2 - T^2\hat{\sigma}_w^2
\end{bmatrix}
\]

(128)

By equalising the matrices obtained from (127) and (128), equations (129) to (132), are obtained.

\[
m_1(1 - g_1) - m_2g_2T = m_1 - 2Tm_2 + T^2m_3 - \frac{T^3}{4}\hat{\sigma}_w^2
\]

(129)

\[
m_2(1 - g_1) - g_1\Delta m_2 = m_2 - Tm_2 + \frac{T^3}{2}\hat{\sigma}_w^2
\]

(130)

\[
m_1 = m_1 - T - Tm_2 + \frac{T^3}{2}\hat{\sigma}_w^2
\]

(131)

\[
-m_2g_1 + m_2(1 - g_1)T = m_2 - T^2\hat{\sigma}_w^2
\]

(132)

By solving (130)–(132), (133)–(135) are achieved.

\[
[I - KH]P_{\text{pre-1}} = \begin{bmatrix}
    m_1(1 - g_1) - m_2g_2T - m_3(1 - g_1) - g_1\Delta m_2 \\
    -g_2m_1 + m_2(1 - g_1)T - g_2m_3 + m_3(1 - g_1)T
\end{bmatrix}
\]

(136)

\[
F^{-1}[P_{\text{pre-1}} - GG^T\hat{\sigma}_w^2](F^{-1})^T =
\begin{bmatrix}
    T^3m_1 + T^2m_3 + T(m_2 + m_3) - 2Tm_2 + m_1 - \frac{T^3}{2}m_3 + \frac{3T^2}{2}m_2 - T(m_1 + m_2) + m_2 - \frac{T^2}{2}m_3 - Tm_2 + m_3 \\
    -T^2m_2 + Tm_3 - m_3 - Tm_2 + m_3
\end{bmatrix}
\]

(137)