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Author's reply

It is claimed that two main results of our paper [2] are incorrect. Here, we prove that the preceding arguments [1] are incorrect, justifying once more, the correctness of the results presented in our paper [2].

1 Preliminaries

To facilitate the material that follows, we briefly present the two main results in question [2]. To this end consider the generalised state space (GSS) system

$$s\tilde{E}X(s) - \tilde{E}x(0_-) = \tilde{A}X(s) + \tilde{B}U(s) \quad Y(s) = \tilde{C}X(s) \tag{1}$$

It is assumed that $\det[s\tilde{E} - \tilde{A}] \not\equiv 0$. For this case, let μ be a real number such that $\det[\mu\tilde{E} + \tilde{A}] \neq 0$. In the special case where $\det \tilde{E} \neq 0$, the system of eqns. 1 is called regular, while in the special case where $\det \tilde{E} = 0$ is called singular. The system may be rewritten as follows:

$$(s + \mu)EX(s) - Ex(0_-) = X(s) + BU(s) \quad Y(s) = \tilde{C}X(s) \tag{2}$$

where $E = (\mu\tilde{E} + \tilde{A})^{-1}\tilde{E}$ and $B = (\mu\tilde{E} + \tilde{A})^{-1}\tilde{B}$. Define

$$\gamma = \text{rank}\,[E \,\vdots\, B] = \text{rank}\,[\tilde{E} \,\vdots\, \tilde{B}]$$

$$\begin{matrix}\gamma\{ \\ n-\gamma\{\end{matrix}\begin{bmatrix} \bar{E} & \vdots & \bar{B} \\ E^+ & \vdots & B^+ \end{bmatrix} = J[EJ^T \,\vdots\, B] \tag{3}$$

$$\begin{matrix}\gamma\{ \\ n-\gamma\{\end{matrix}\begin{bmatrix} V(s) \\ V^+(s) \end{bmatrix} = JX(s)$$

$$\begin{matrix}\gamma\{ \\ n-\gamma\{\end{matrix}\begin{bmatrix} v(0_-) \\ v^+(0_-) \end{bmatrix} = Jx(0_-); \; J = \begin{bmatrix} J_1 \\ J_2 \end{bmatrix}\begin{matrix}\}\gamma \\ \}n-\gamma\end{matrix} \tag{4}$$

where J is an $n \times n$ rearrangement matrix, where its submatrix J_1 selects the linearly independent rows of $[E \,\vdots\, B]$ and J_2 selects the linearly dependent rows of $[E \,\vdots\, B]$. The matrices $[E^+ \,\vdots\, B^+] = J_2[EJ^T \,\vdots\, B]$ and $[\bar{E} \,\vdots\, \bar{B}] = J_1[EJ^T \,\vdots\, B]$ are related as follows: $J_2[EJ^T \,\vdots\, B] = QJ_1[EJ^T \,\vdots\, B]$, where $Q = [J_2EE^TJ_1 + J_2BB^TJ_1][J_1EE^TJ_1 + J_1BB^TJ_1]^{-1}$. Applying the transformation eqn. 4 to the system of eqns. 2, we readily derive the following description (equivalent to eqns. 2):

$$(s + \mu)AV(s) = V(s) + \bar{B}U(s) + Av(0_-) + \hat{E}v^{\#}(0_-)$$

$$Y(s) = CV(s) \tag{5a}$$

$$V^+(s) = QV(s) \tag{5b}$$

where

$$A = J_1EJ^T\begin{bmatrix} I_\gamma \\ Q \end{bmatrix} \quad \hat{E} = J_1EJ^T\begin{bmatrix} 0 \\ -I_{n-\gamma} \end{bmatrix}$$

$$C = \tilde{C}J^T\begin{bmatrix} I_\gamma \\ Q \end{bmatrix} \quad y^{\#}(0_-) = Qv(0_-) - v^+(0) \tag{6}$$

The proposed observer is of the type

$$sZ(s) - z(0_-) = FZ(s) + \hat{G}Y(s) + LU(s) \tag{7a}$$

where the estimate of $v(t)$, denoted by $v_e(t)$, is given by the relation

$$\begin{bmatrix} z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} T \\ C \end{bmatrix}v_e(t); \; \text{rank}\begin{bmatrix} T \\ C \end{bmatrix} = \gamma \tag{7b}$$

where, without loss of generality, we have assumed that $\text{rank}\, C = p$ and where F, $\hat{G}$, L and T are appropriate matrices specified by the designer. The estimate $x_e(t)$ of the original state vector $x(t)$ is given in terms of $v_e(t)$ by the relation

$$x_e(t) = J^T\begin{bmatrix} I \\ Q \end{bmatrix}v_e(t) \tag{7c}$$

Eqns. 7*b* and *c* reveal that for the present case the estimate of the state vector is given as a linear map of the output vector y and the observer's state vector z. This type of observer is the type proposed by Luenberger in References 6–8 for the case of regular systems. This type of observer is also used in References 9, 10 and 11 for the case of singular systems. An observer of a familiar type to that of eqns. 7 may be found in References 12, 13 and 16, wherein the request of trajectory matching has been replaced by a trajectory following requirement.

The two main results in question are the following:

(*a*) In Theorem 3.1 [2] it was proven that: An observer of the eqns. 7, type with arbitrary eigenvalues, can be designed for the linear time-invariant singular system in eqn. 1, if and only if one of the two equivalent criteria holds:

(i) the system of eqn. 5*a*, with $v^{\#}(0_-) = 0$, is observable

(ii) the system of eqn. 1 is modal observable, and

$$\text{rank}\left\{\begin{bmatrix} \tilde{E} \\ \tilde{C} \end{bmatrix}J^T\begin{bmatrix} I_\gamma \\ Q \end{bmatrix}\right\} = \gamma \tag{8}$$

(*b*) Also it has been proven in Reference 2 that the order of the minimal observer is $\gamma - \text{rank}\, C$.

2 Correctness of results

In Reference 1 the following four arguments have developed to establish the incorrectness of the results in Reference 2:

(*a*) The regularity of $[s\tilde{E} - \tilde{A}]$ can be relaxed as already done in References 3 and 4.

(*b*) The necessary and sufficient conditions presented in Reference 2 are too restrictive, since in References 3 and 5 an observer can be designed under less restrictive conditions (see also examples 1 and 2 in Reference 1).

(*c*) The resulting observer in Reference 5 is of order $\text{rank}\, A - d$, where d is the rank of a submatrix of the equivalent output matrix. The order of the resulting observer in Reference 3 is lower. Both orders are less than $\gamma - \text{rank}\, C$, i.e, they are less than the order of the minimal observer proven in Reference 2.

(*d*) Other papers in the field used the observability of the eqn. 5*a* system with $\boldsymbol{v}^{\#}(0_-) = \mathbf{0}$, as a sufficient condition and not as a necessary one [9, 10, 12–19].

With regard to argument (*a*), we mention that the condition $\det[s\tilde{\boldsymbol{E}} - \tilde{\boldsymbol{A}}] \not\equiv 0$ characterises the class of GSS systems (see for example References 5, 20–22). The case where the matrix $s\tilde{\boldsymbol{E}} - \tilde{\boldsymbol{A}}$ is not necessarily regular, characterises another class of systems, namely implicit systems [23]. Clearly, the regularity assumption cannot establish any incorrectness. However, as far as we know, the results regarding the observer design for implicit systems are limited. In Reference 3, only sufficient conditions are derived, while in Reference 4, the problem for the discrete time case is reduced to the respective GSS problem only for special cases.

With regard to argument (*b*), we mention that the conditions presented in References 3 and 5 refer to an observer type different from that proposed in Reference 2. The observer used in References 3 and 5 has the distinct characteristic in that the estimate $\boldsymbol{x}_e$ of the state vector is expressed not only as a linear map of the output vector y and the observer's state vector z, but also as a linear map of the input vector $\boldsymbol{u}$, i.e. it is expressed as

$$x_e(t) = \boldsymbol{H}_1 y(t) + \boldsymbol{H}_2 z(t) + \boldsymbol{H}_3 \boldsymbol{u}(t) \quad (9)$$

For convenience, in what follows, the observers of eqns. 7 and 9 will be called observers of type 1 and 2, respectively. These two types of observers are fundamentally different. The fact that the type of observer used in Reference 2 is of type 1 way remarked repeatedly Reference 2 (for example, in Preliminaries and Definitions 3.1, 3.2 and 3.3). Clearly, by using different types of observers, one arrives at different necessary and sufficient conditions for the problem to have a solution. Therefore, the two results in question are not comparable for the observers of type 1 and 2 in the sense that their difference cannot establish any incorrectness. A brief discussion of the advantages and disadvantages of the different types of observers proposed for GSS systems follows in Section 4.

With regard to argument (*c*), an analogous reasoning to that for argument 2 can be readily developed. It is important to mention that, in general, the minimal order of an observer of type 2 will be less than the minimal order of an observer of type 1. However, it is also important that the minimal order observer derived in Reference 5 is not of order rank $A - d$ (as claimed in Reference 1), but of the greater order rank $\boldsymbol{E} - d$.

Argument (*d*) is based on an incorrect observation. It is incorrect that the papers of References 9, 10, 12–19 have assumed the observability of the system of eqn. 5*a*, with $\boldsymbol{v}^{\#}(0_-) = \mathbf{0}$, as a sufficient condition; instead the observability of the system of eqn. 1 has been assumed. The system of eqn. 5*a*, with $\boldsymbol{v}^{\#}(0_-) = \mathbf{0}$, is not equivalent to that in eqn. 1. In Appendix B of Reference 2, it has been proven in detail that the observability of the eqn. 5*a* system with $\boldsymbol{v}^{\#}(0_-) = \mathbf{0}$, is a wider condition than the observability of eqn. 1 system.

4 Selection of observer type

Even though the advantages of an observer of type 2 are significant (they cover a wider class of GSS systems and the order of the minimal observer is lower), the selection of the observer type may be further constrained by other practical requirements. This is why both types of observers have been studied in the literature (see Reference 9–13, 16 for type 1, and 3, 13–15 for type 2).

In brief, we denote that an observer of type 1 operates as a low-pass filter of the input signal $\boldsymbol{u}$, thus influencing beneficially the feedback of the state vector's estimate, in cases where the input is noisy. This characteristic is rather important, since for GSS systems derivative feedback of the state vector is commonly used [24, 25].

Furthermore, we denote that the wide use of the observer design technique is mainly due to its robust characteristics in cases where the system data are perturbed. These characteristics result from the robust characteristics of the pole assignment problem, to which the observer design problem is essentially reduced. As the number of the state components estimated via a pole assignment procedure is increased, the robustness of the overall design is further improved. This is why, in many practical applications, even full-order observers become preferable.

In the example that follows in Section 5, the advantages of an observer type 1 over those of an observer type 2 are demonstrated for a GSS plant, involving small variations in its parameters. In the same example, the advantages of observers of type 1 in the unification of the observer design between regular and singular systems, are also illustrated.

It is important to mention that except of types 1 and 2, other types of observers have also been proposed in the literature. We focus our attention to one more type: the case where the observer dynamics are described as a singular system [13 and 16]. Clearly, this type appear to have many disadvantages mainly due to its improper part yielding bad influence to the numerical and/or signal noise.

A more detailed and complete investigation of the above aspects as well as other, related issues may be found in Reference 26.

5 Example

Consider the GSS system:

$$\left.\begin{aligned} \dot{x}_1(t) &= x_1(t) + x_2(t) + u(t) \\ 0 &= x_2(t) + u(t) \\ y(t) &= 2x_1(t) + x_2(t) \end{aligned}\right\} \quad (10)$$

An observer type 1 is of the form

$$\left.\begin{aligned} \dot{z}(t) &= fz(t) + (1-f)[y(t) + u(t)] \\ x_{1e}(t) &= (1/2)z(t) \\ x_{2e}(t) &= y(t) - z(t) \end{aligned}\right\} \quad (11)$$

where x_{1e} and x_{2e} denote the estimates of the state variables x_1 and x_2, respectively. An observer type 2, for the system of eqns. 10, is of the form

$$\left.\begin{aligned} x_{1e}(t) &= (1/2)y(t) + (1/2)u(t) \\ x_{2e}(t) &= -u(t) \end{aligned}\right\} \quad (12)$$

It is possible for the algebraic equation appearing in eqns. 10 to be perturbed for a sort period of time τ by an external event in such a way that for this period of time, i.e. $t \in [T, T + \tau]$, eqns. 10 take the form

$$\left.\begin{aligned} \dot{x}_1(t) &= x_1(t) + x_2(t) + u(t) \\ \varepsilon\dot{x}_2(t) &= x_2(t) + u(t) \\ y(t) &= 2x_1(t) + x_2(t) \end{aligned}\right\} \quad (13)$$

where ε denotes a very small positive perturbation ($\varepsilon \ll 1$). Usually, and to have a finite energy external event, the time distance depends on the perturbation ε, for example it may be of the form $\tau = \varepsilon^{-2}$. Interchanges between eqns. 10 and 13 often take place in practical GSS systems. However, GSS systems cover singularly perturbed systems as a special case. It is clear that 'for many applications, it is useful to have a theory of control which treats regular systems and singular systems together in a unified way, rather than separately' [27].

Applying the observer types of eqns. 11 and 12 to the system of eqns. 13 for $u(t) = 1$, we derive the following estimation errors. For the observer type 1:

$$\left.\begin{aligned} e_1(t) &= \exp\{f(t-T)\}\left[e_1(T) + \left(\frac{c^*}{2}\right)\left(\frac{(1+f)\varepsilon}{f\varepsilon - 1}\right)\right] \\ &\quad - \exp\{\varepsilon^{-1}(t-T)\}\left(\frac{c^*}{2}\right)\left[\frac{(1+f)\varepsilon}{f\varepsilon - 1}\right] \\ e_2(t) &= -2\exp\{f(t-T)\}\left[e_1(T) + \left(\frac{c^*}{2}\right)\left(\frac{(1+f)\varepsilon}{f\varepsilon - 1}\right)\right] \\ &\quad - 2\exp\{\varepsilon^{-1}(t-T)\}\left(\frac{c^*}{2}\right)\left[\frac{(1+f)\varepsilon}{f\varepsilon - 1}\right] \end{aligned}\right\} \tag{14}$$

where $e_1(t) = x_{1e}(t) - x_1(t)$, $e_2(t) = x_{2e}(t) - x_2(t)$ and $c^* = u(T) + x_2(T)$. For the observer type 2:

$$\left.\begin{aligned} e_1(t) &= \exp\{\varepsilon^{-1}(t-T)\}\left(\frac{c^*}{2}\right) \\ e_2(t) &= -c^*\exp\{\varepsilon^{-1}(t-T)\} \end{aligned}\right\} \tag{15}$$

For the observer type 1, and if we let f to be a negative number with large enough norm (i.e. to have $\exp\{f(t-T)\} \simeq 0$), we have

$$\lim_{\varepsilon \to 0}\{e_1(t)\} \cong 0 \quad \lim_{\varepsilon \to 0}\{e_2(t)\} \cong 0$$

For the observer type 2 we have

$$\lim_{\varepsilon \to 0}\{e_1(t)\} = c^*/2 \quad \lim_{\varepsilon \to 0}\{e_2(t)\} = -c^*$$

Clearly, observer type 2 appears to be inappropriate to cover the perturbed case. Observer type 1 facilitates the unified treatment of both cases, since it has the distinct characteristic in that it unifies the solution of the observer design problem for regular and singular systems.

6 Conclusion

In closing, we thank the authors of the preceding note for their interest in our observer design approach. Although their remarks appear inaccurate, nevertheless they have offered us an opportunity to clarify some points regarding the different types of observers used for GSS systems.

P.N. PARASKEVOPOULOS *10th August, 1992*
F.N. KOUMBOULIS

Department of Electrical Engineering
National Technical University of Athens
15773 Zographou Athens
Greece

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